Chapter 6

# Quantum Noise in Optical Amplifiers

Kyo Inoue

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**Abstract**

Noise is one of the basic characteristics of optical amplifiers. Whereas there are various noise sources, the intrinsic one is quantum noise that originates from Heisenberg's uncertainty principle. This chapter describes quantum noise in optical amplifiers, including population-inversion–based amplifiers such as an Erbium-doped fiber amplifier and a semiconductor optical amplifier, and optical parametric amplifiers. A full quantum mechanical treatment is developed based on Heisenberg equation of motion for quantum mechanical operators. The results provide the quantum mechanical basis for a classical picture of amplifier noise widely used in the optical communication field.

**Keywords:** quantum noise, noise figure, quantum mechanics, population-inversion–based amplifier, optical parametric amplifier

## 1. Introduction

Noise is one of the important properties in optical amplifiers [1]. The intrinsic noise characteristic is determined by quantum mechanics, especially Heisenberg's uncertainty principle. This chapter describes quantum noise in optical amplifiers in terms of quantum mechanics. After brief introduction of a classical treatment usually used in the optical communication field, properties of an optically amplified light, such as the mean amplitude, the mean photon number, and their variances, are derived based on first principles of quantum mechanics. Two kinds of optical amplifiers are treated: amplifiers based on two-level interaction in a population-inverted medium, i.e., an Erbium-doped fiber amplifier and a semiconductor optical amplifier, and those based on parametric interaction in an optical nonlinear medium. The results presented here provide the quantum mechanical basis to a phenomenological classical treatment conventionally employed for describing amplifier noise.

## 2. Classical treatment

A classical treatment of amplifier noise is widely employed in the optical communication field [1, 2], whereas it originates from quantum mechanics. Before presenting a quantum mechanical treatment, we briefly introduce the classical treatment. We first consider the light intensity or the photon number outputted from an amplifier. A photon-number rate equation for light propagating through a population-inverted medium can be expressed as

$$\frac{dn}{dz} = gN_2n - gN_1n + gN_2, \tag{1}$$

where $n$ is the number of photons, $N_2$ and $N_1$ are the numbers of atoms at the higher and lower energy states in the medium, respectively, and $g$ is a constant representing the photon emission/absorption efficiency. The first, second, and third terms represent stimulated emission, absorption, and spontaneous emission, respectively. The efficiency $g$ is common in this phenomenon in a simple two-level model [3]. Assuming that $N_1$ and $N_2$ are uniform along the medium length, the photon number at the output is calculated from Eq. (1) as

$$n_{\text{out}} = n_{\text{in}}e^{g(N_2-N_1)L} + \frac{N_2}{N_2 - N_1}\left\{e^{g(N_2-N_1)L} - 1\right\}, \tag{2}$$

where $L$ is the medium length. The first term represents amplified signal photons with a signal gain of $\exp[g(N_2 - N_1)L] \equiv G$. The second term represents amplified spontaneous emission (ASE) photons, which can be rewritten as $n_{\text{sp}}(G - 1)$ with $n_{\text{sp}} \equiv N_2/(N_2 - N_1)$. The parameter $n_{\text{sp}}$ is called "population inversion parameter" or "noise factor." Note that the above equations are for the photon number in one mode in terms of the frequency and the polarization.

Eq. (2) shows that the output photon number is composed of amplified signal photons and ASE photons. Accordingly, the output amplitude is supposed to be a summation of amplitudes of amplified signal and ASE lights as

$$E_{\text{out}} = \sqrt{G}E_{\text{in}} + E_{\text{ASE}}, \tag{3}$$

where $E_{\text{out}}$, $E_{\text{in}}$, and $E_{\text{ASE}}$ are the amplitudes of the output light, the input signal light, and ASE light, respectively. The second term provides the ASE power as $<|E_{\text{ASE}}|^2> = n_{\text{sp}}(G - 1)hf\Delta f$, where $< >$ denotes the mean value, $hf$ is one photon energy, and $\Delta f$ is the ASE bandwidth. Regarding the ASE phase, it is supposed to be completely random because spontaneous emission occurs randomly. Thus, the average of the ASE amplitude is supposed to be zero: $<E_{\text{ASE}}> = 0$. Here, we decompose $E_{\text{ASE}}$ into the real and imaginary parts, which are supposed to be isotropic because the phase is random: $<\{\text{Re}[E_{\text{ASE}}]\}^2> = <\{\text{Im}[E_{\text{ASE}}]\}^2> = <|E_{\text{ASE}}|^2>/2 = n_{\text{sp}}(G -1)hf\Delta f/2$. Subsequently, the variance of each amplitude component is $<\{\text{Re}[E_{\text{ASE}}]\}^2> - <\text{Re}[E_{\text{ASE}}]>^2 = <\{\text{Im}[E_{\text{ASE}}]\}^2> - <\text{Im}[E_{\text{ASE}}]>^2 = n_{\text{sp}}(G - 1)hf\Delta f/2$.

Intensity noise is evaluated using Eq. (3). The output intensity is given by $I_{\text{out}} = |E_{\text{out}}|^2$, and its fluctuation is evaluated by the variance of $I_{\text{out}}$ as

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$$< I_{out}^2 > - < I_{out}>^2 =< |E_{out}|^4 > - < |E_{out}|^2>^2 = 2G|E_{in}|^2 < |E_{ASE}|^2 > + < |E_{ASE}|^4 > - < |E_{ASE}|^2>^2, \tag{4}$$

where the postulate of the ASE light phase being random is used in averaging. The first term represents 2 × <signal output intensity> × <ASE intensity>, which is called the "signal-spontaneous beat noise." The second and third terms represent the intensity variance of the ASE light, which is called the "spontaneous-spontaneous beat noise."

As an indicator for the amplifier noise performance, the "noise figure (NF)" is usually used. It is defined as the ratio of the signal-to-noise ratios (SNRs) at the input and output of an amplifier in terms of the optical intensity: NF ≡ $(\mathrm{SNR})_{in}/(\mathrm{SNR})_{out}$ where SNR ≡ (mean intensity)$^2$/(variance of the intensity) in the signal mode. The square of the mean intensity at the output is calculated from Eq. (3) as $<|E_{out}|^2> = G|E_{in}|^2 + <|E_{ASE}|^2>$, and the output variance is expressed as Eq. (4); thus the output SNR is expressed as

$$(\mathrm{SNR})_{out} = \frac{G^2|E_{in}|^2 + 2G|E_{in}|^2 < |E_{ASE}|^2 > + < |E_{ASE}|^2>^2}{2G|E_{in}|^2 < |E_{ASE}|^2 > + < |E_{ASE}|^4 > - < |E_{ASE}|^2>^2}. \tag{5}$$

On the other hand, the input SNR is evaluated for pure monochromatic light in the definition of the noise figure. In quantum mechanics, such a light is called "coherent state," whose photon-number variance is equal to the mean photon number: $<n^2> - <n>^2 = <n>$. Thus, $<|E_{in}|^4> - <|E_{in}|^2>^2 = |E_{in}|^2 hf$ where $hf$ is attached for the dimension to be matched. Subsequently, the input SNR is given by $<|E_{in}|^2>^2/(<|E_{in}|^4> - <|E_{in}|^2>^2) = |E_{in}|^2/hf$. Therefore, the NF is expressed as

$$\mathrm{NF} = \frac{(\mathrm{SNR})_{in}}{(\mathrm{SNR})_{out}} \approx \frac{|E_{in}|^2}{hf} \cdot \frac{2G|E_{in}|^2 < |E_{ASE}|^2 >}{G^2|E_{in}|^4} = 2n_{sp}\frac{G-1}{G}, \tag{6}$$

where $G|E_{in}|^2 >> |E_{ASE}|^2$ is assumed, and $<|E_{ASE}|^2> = n_{sp}(G-1)hf$ is substituted because the signal mode is considered here.

The above-mentioned classical treatment is widely used for noise in optical amplifiers. However, it is based on phenomenological assumptions. (i) Eq. (3) is phenomenologically provided. Though the solution of the photon-number rate equation indicates that the output photon number is composed of amplified signal photons and ASE photons (Eq. (2)), this result does not logically conclude that the output amplitude is a linear summation of the amplified signal and the ASE amplitudes as Eq. (3). (ii) The phase of ASE light is assumed to be random, which is a phenomenological postulate, not logically derived from first principles. Although the above classical treatment is correct and useful in fact, we need quantum mechanics for theoretically confirming its validity, which is presented in the following sections.

## 3. Quantum mechanics

In this section, we briefly review quantum mechanics, especially the Heisenberg picture [4]. The basic concept of quantum mechanics is that a physical state is probabilistic and the theory only provides mean values of physical quantities, which is given by a quantum mechanical inner product of a physical quantity operator $\widehat{x}$ with respect to an objective state $|\Psi>$ as $< \Psi|\widehat{x}|\Psi >$. In the Heisenberg picture, an operator evolves according to the Heisenberg equation of motion as

$$\frac{d\widehat{x}}{dt} = \frac{1}{i\hbar}\left[\widehat{x}, \widehat{H}\right], \tag{7}$$

where $\widehat{H}$ is the Hamiltonian (i.e., the energy operator) of a concerned system, $[\widehat{x}, \widehat{y}] = \widehat{x}\widehat{y} - \widehat{y}\widehat{x}$ denotes a commutator, and $\hbar$ is Planck's constant. The mean value after the evolution is given by the inner product $< \Psi_0|\widehat{x}(t)|\Psi_0 >$ where $|\Psi_0>$ is an initial state.

The most important operator in discussing quantum mechanical properties of light is the "annihilation operator," $\widehat{a}$, which corresponds to the complex amplitude of light and is also called as the "field operator." It has an eigenstate $|\alpha>$ and an eigenvalue $\alpha$:$\widehat{a}|\alpha >= \alpha|\alpha >$. This eigenstate $|\alpha>$ is called "coherent state" and corresponds to pure monochromatic light in classical optics, and the eigenvalue $\alpha$ corresponds to its complex amplitude in the photon-number unit. The Hermitian conjugate of the annihilation operator $\widehat{a}^{\dagger}$ is called the "creation operator," which satisfies $< \alpha|\widehat{a}^{\dagger} = \alpha^* < \alpha|$. The annihilation and creation operators have a commutation relationship $\left[\widehat{a}, \widehat{a}^{\dagger}\right] = 1$, which comes from Heisenberg's uncertainty principle and is the origin of quantum noise. They also work to annihilate or create a photon as $\widehat{a}|n >= \sqrt{n}|n-1>$ and $\widehat{a}^{\dagger}|n >= \sqrt{n+1}|n+1>$, where $|n>$ is a photon number state having $n$ photons. Another important operator is the photon-number operator defined as $\widehat{n} \equiv \widehat{a}^{\dagger}\widehat{a}$. Using the above eigenvalue/eigenstate equations, the mean photon number of a coherent state is given by $< \alpha|\widehat{n}|\alpha >=< \alpha|\widehat{a}^{\dagger}\widehat{a}|\alpha >= \alpha^*\alpha < \alpha|\alpha >= |\alpha|^2$, which corresponds to the classical picture that the absolute square of a complex amplitude represents the light intensity.

We discuss quantum noise of optical amplifiers in the following sections, using the above-mentioned framework of quantum mechanics. Note that the above operator $\widehat{a}$ is for one mode in terms of the frequency and the polarization state. Thus, the following discussions are for one optical mode.

## 4. Population-inversion–based amplifiers

Erbium-doped fiber amplifiers (EDFAs) are widely used in optical communications. Optical semiconductor amplifiers are also being developed for compact and integrated amplifying devices. They amplify signal light through interaction between light and a two-level atomic system with population inversion. This section discusses quantum noise in population-inversion-based amplifiers [5].

### 4.1. Heisenberg equation

The Hamiltonian for a light-atom interacting system can be expressed as [4]

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$$\widehat{H} = \hbar\omega\widehat{a}^{\dagger}\widehat{a} + \sum_{j} \hbar\omega_{\mathrm{a}}^{(j)}\widehat{\pi}_{j}^{\dagger}\widehat{\pi}_{j} + i\hbar\sum_{j} \kappa_{j}\left(\widehat{\pi}_{j}^{\dagger}\widehat{a} - \widehat{\pi}_{j}\widehat{a}^{\dagger}\right). \tag{8}$$

The first and second terms are the Hamiltonians of light and atoms without interaction, respectively, where $\widehat{a}$ is the field operator; $\omega$ is the angular frequency of light; $\widehat{\pi}_j^{\dagger} = |2 >< 1|_j$ and $\widehat{\pi}_j = |1 >< 2|_j$ are the transition operators of an atom, with $|2>_j$ and $|1>_j$ denoting the upper and lower energy states, respectively, satisfying $<1|1>_{j,k} = <2|2>_{j,k} = \delta_{j,k}$ and $<1|2>_{j,k} = <2|1>_{j,k} = 0$; $\hbar\omega_{\mathrm{a}}^{(j)}$ is the energy difference between the upper and lower states; the subscript $j$ indicates a specific atom. The third term is the interaction Hamiltonian between light and atoms, which represents energy exchange such that a photon is created while an atom transits from the upper to the lower states and vice versa, with $\kappa_j$ being the coupling coefficient.

Applying the above Hamiltonian to the Heisenberg equations for $\widehat{a}$ and $\widehat{\pi}_j$, we obtain the following differential equations:

$$\frac{d\widehat{a}}{dt} = \frac{1}{i\hbar}\left[\widehat{a}, \widehat{H}\right] = -i\omega\widehat{a} - \sum_{j} \kappa_j\widehat{\pi}_j, \tag{9a}$$

$$\frac{d\widehat{\pi}_j}{dt} = \frac{1}{i\hbar}\left[\widehat{\pi}_j, \widehat{H}\right] = -i\omega_{\mathrm{a}}^{(j)}\widehat{\pi}_j + \kappa_j\widehat{a}\left(\widehat{\pi}_j\widehat{\pi}_j^{\dagger} - \widehat{\pi}_j^{\dagger}\widehat{\pi}_j\right). \tag{9b}$$

Employing the variable translations $\widehat{a} \to \widehat{a}\exp(-i\omega t)$ and $\widehat{\pi}_j \to \widehat{\pi}_j\exp\left(-i\omega_{\mathrm{a}}^{(j)}t\right)$, these equations are rewritten as

$$\frac{d\widehat{a}}{dt} = -\sum_{j} \kappa_j\widehat{\pi}_j e^{i\Delta\omega_j t}, \tag{10a}$$

$$\frac{d\widehat{\pi}_j}{dt} = \kappa_j\widehat{a}\left(\widehat{\pi}_j\widehat{\pi}_j^{\dagger} - \widehat{\pi}_j^{\dagger}\widehat{\pi}_j\right)e^{-i\Delta\omega_j t}, \tag{10b}$$

where $\Delta\omega_j \equiv \omega - \omega_{\mathrm{a}}^{(j)}$, i.e., the frequency detuning between light and a two-level system.

We solve Eq. (10) by an iterative approximation. First, the first-order solutions are derived by substituting the initial values $\{\widehat{a}^{(0)}, \widehat{\pi}_j^{(0)}\}$ into the right-hand side of the equations:

$$\frac{d\widehat{a}^{(1)}}{dt} = -\sum_{j} \kappa_j\widehat{\pi}_j^{(0)} e^{i\Delta\omega_j t}, \tag{11a}$$

$$\frac{d\widehat{\pi}_j^{(1)}}{dt} = \kappa_j\widehat{a}^{(0)}\left(\widehat{\pi}_j^{(0)}\widehat{\pi}_j^{(0)\dagger} - \widehat{\pi}_j^{(0)\dagger}\widehat{\pi}_j^{(0)}\right)e^{-i\Delta\omega_j t}. \tag{11b}$$

The solutions of these equations are

$$\widehat{a}^{(1)} = \widehat{a}^{(0)} + i\sum_{j} \kappa_j\widehat{\pi}_j^{(0)}\frac{e^{i\Delta\omega_j t} - 1}{\Delta\omega_j}, \tag{12a}$$

$$\widehat{\pi}_j^{(1)} = \widehat{\pi}_j^{(0)} + i\kappa_j \widehat{a}^{(0)} \left(\widehat{\pi}_j^{(0)} \widehat{\pi}_j^{(0)\dagger} - \widehat{\pi}_j^{(0)\dagger} \widehat{\pi}_j^{(0)}\right) \frac{e^{-i\Delta\omega_j t} - 1}{\Delta\omega_j}. \tag{12b}$$

Next, these first-order solutions are substituted into the right-hand side of Eq. (10a), and the second-order solution is calculated as

$$\widehat{a}^{(2)} = \widehat{a}^{(0)} \left\{ 1 - \sum_j \kappa_j^2 \left(\widehat{\pi}_j^{(0)} \widehat{\pi}_j^{(0)\dagger} - \widehat{\pi}_j^{(0)\dagger} \widehat{\pi}_j^{(0)}\right) \frac{1 - e^{i\Delta\omega_j t} + i\Delta\omega_j t}{\left(\Delta\omega_j\right)^2} \right\} + i \sum_j \kappa_j \widehat{\pi}_j^{(0)} \frac{e^{i\Delta\omega_j t} - 1}{\Delta\omega_j}. \tag{13}$$

We regard Eq. (13) as the time evolution of the field operator during a short time $\tau$, and rewrite it as

$$\widehat{a}(t_0 + \tau) = \widehat{a}(t_0) \left\{ 1 - \widehat{\Pi}(t_0) \right\} + \widehat{\mathrm{P}}(t_0), \tag{14}$$

where

$$\widehat{\Pi}(t_0) = \sum_j \kappa_j^2 \left\{ \widehat{\pi}_j(t_0) \widehat{\pi}_j^\dagger(t_0) - \widehat{\pi}_j^\dagger(t_0) \widehat{\pi}_j(t_0) \right\} \frac{2\sin^2\left(\Delta\omega_j \tau/2\right) + i\{\Delta\omega_j \tau - \sin\left(\Delta\omega_j \tau\right)\}}{\left(\Delta\omega_j\right)^2}, \tag{15a}$$

$$\widehat{\mathrm{P}}(t_0) = i \sum_j \kappa_j \widehat{\pi}_j(t_0) \frac{e^{i\Delta\omega_j \tau} - 1}{\Delta\omega_j}. \tag{15b}$$

Eq. (14) is the basic expression for discussing quantum properties of light that travels through an amplifier. For the discussion, we also need an initial state of the system at $t_0$. It can be expressed as

$$|\Psi_0 >= |\Psi(t_0) > \otimes |\Psi_{\mathrm{a}}(t_0) >= |\Psi(t_0) > \otimes \left\{ \bigotimes_j \left(c_1|1 > + c_2|2 >\right)_j \right\}, \tag{16}$$

where $|\Psi(t_0)>$ denotes the initial state of light, $|\Psi_{\mathrm{a}}(t_0)>$ denotes that of atoms, and $c_1$ and $c_2$ are the probability amplitudes of an atom being in the lower and upper states, respectively, satisfying $|c_1|^2 + |c_2|^2 = 1$. We use Eqs. (14) and (16) in the following calculations.

### 4.2. Mean amplitude

We first discuss the mean amplitude. The mean amplitude, denoted as $\bar{a}$ hereafter, after a short-time interaction is expressed from Eq. (14) as

$$\begin{aligned} \bar{a}(t_0 + \tau) &=< \Psi_0|\widehat{a}\,(t_0 + \tau)|\Psi_0 >=< \Psi(t_0)|\widehat{a}(t_0 + \tau)|\Psi(t_0) > \\ &\times \left\{ 1 - < \Psi_{\mathrm{a}}(t_0)|\widehat{\Pi}(t_0)|\Psi_{\mathrm{a}}(t_0) > \right\} + < \Psi_{\mathrm{a}}(t_0)|\widehat{\mathrm{P}}(t_0)|\Psi_{\mathrm{a}}(t_0) > . \end{aligned} \tag{17}$$

The average of the transition operator $\widehat{\Pi}$ is calculated, using Eqs. (15a) and (16), as

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$$< \Psi_a(t_0)|\widehat{\Pi}(t_0)|\Psi_a(t_0) >= \sum_j \kappa_j^2 \left(|c_1|^2 - |c_2|^2\right)_j \frac{2\sin^2(\Delta\omega_j\tau/2) + i\{\Delta\omega_j\tau - \sin(\Delta\omega_j\tau)\}}{(\Delta\omega_j)^2}. \quad (18)$$

Assuming that the energy levels of each atom are densely distributed, the summation in this equation can be replaced by an integral in the frequency domain, and the real part of Eq. (18) is further calculated as

$$\begin{aligned}\sum_j \kappa_j^2 \left(|c_1|^2 - |c_2|^2\right)_j \frac{2\sin^2(\Delta\omega_j\tau/2)}{(\Delta\omega_j)^2} &= \int_{-\infty}^{\infty} \kappa^2 \left(|c_1|^2 - |c_2|^2\right)_\Omega \frac{2\sin^2(\Omega\tau/2)}{\Omega^2}\rho(\Omega)d\Omega \\ &= 2\kappa^2\left(|c_1|^2 - |c_2|^2\right)\int_{-\infty}^{\infty}\frac{\sin^2(\Omega\tau/2)}{\Omega^2}\rho(\Omega)d\Omega = \kappa^2(N_1 - N_2)\pi\tau,\end{aligned} \quad (19)$$

where $\Omega$ is the frequency detuning; $\rho$ is the density of atoms in the frequency domain and is assumed to be constant around a resonant frequency $\Omega = 0$ as $\rho_0$; $\{\kappa, |c_2|^2$, and $|c_1|^2\}$ are assumed to be identical for any atom; and $N_1 \equiv \rho_0|c_1|^2$ and $N_2 \equiv \rho_0|c_2|^2$ are the numbers of atoms at the lower and upper energy states, respectively. On the other hand, the imaginary part of Eq. (18) can be rewritten as

$$\sum_j \kappa_j^2 \left(|c_1|^2 - |c_2|^2\right)_j \frac{\Delta\omega_j\tau - \sin(\Delta\omega_j\tau)}{(\Delta\omega_j)^2} = \int_{-\infty}^{\infty} \kappa^2 \left(|c_1|^2 - |c_2|^2\right)_\Omega \frac{\Omega\tau - \sin(\Omega\tau)}{\Omega^2}\rho(\Omega)d\Omega. \quad (20)$$

In this expression, the contents of the integral is an odd function around the resonant frequency $\Omega = 0$. Thus, the imaginary part equals 0. Regarding the average of $\widehat{\mathrm{P}}$ in Eq. (17), it is calculated as $< \Psi_a(t_0)\left|\widehat{\mathrm{P}}\right|\Psi_a(t_0) >= i\sum_j \kappa_j\left(c_1^* c_2\right)_j\{\exp(i\Delta\omega_j\tau) - 1\}/\Delta\omega_j$. Here, the phases of the probability amplitudes are random in general, and $(c_1^* c_2)_j = 0$ on average. Subsequently, $< \Psi_a(t_0)\left|\widehat{\mathrm{P}}\right|\Psi_a(t_0) >= 0$. Substituting the averages evaluated as above, Eq. (17) can be rewritten as

$$\overline{a}(t_0 + \tau) = \overline{a}(t_0)\{1 + \kappa^2(N_2 - N_1)\pi\tau\}. \quad (21)$$

Eq. (21) describes the time evolution of the mean amplitude of light traveling through an amplifying medium, i.e., the time evolution in a frame moving along with the light, during a short time. This expression can be translated to the spatial evolution along the medium length as

$$\overline{a}(z_0 + \Delta z) = \overline{a}(z_0)\{1 + (g/2)(N_2 - N_1)\Delta z\}, \quad (22)$$

where $g \equiv 2\kappa^2\pi/v$ with $v$ being the light velocity. Applying a Taylor expansion $x(z_0 + \Delta z) = x(z_0) + [dx/dz](z_0) \times \Delta z$ to this equation, we obtain the following differential equation:

$$\frac{d\overline{a}}{dz} = \frac{g}{2}(N_2 - N_1)\overline{a}. \quad (23)$$

Eq. (23) includes ($N_2 - N_1$), which depends on the atoms' state at a local position. Here, we assume that the atoms' state is uniform along the medium length independent of $z$, and this condition is satisfied in uniformly pumped amplifiers with no gain saturation. With this assumption, Eq. (23) can be analytically solved, and the mean amplitude at the amplifier output is expressed as

$$\bar{a}_{\text{out}} = \sqrt{G}\bar{a}_{\text{in}}, \tag{24}$$

where $G \equiv \exp[g(N_2 - N_1)L]$ with $L$ being the amplifier length. This result, derived from the Heisenberg equation, is equivalent to the classical expression, i.e., Eq. (3) with $<E_{\text{ASE}}> = 0$, and confirms that the mean amplitude of ASE light is zero.

### 4.3. Mean photon number

We next discuss the mean photon number. The short-time evolution of the photon-number operator is expressed from Eq. (14) as

$$\hat{n}(t_0+\tau) = \left[\hat{a}^{\dagger}(t_0)\left\{1-\hat{\Pi}^{\dagger}(t_0)\right\}+\hat{\mathrm{P}}^{\dagger}(t_0)\right]\left[\hat{a}(t_0)\left\{1-\hat{\Pi}(t_0)\right\}+\hat{\mathrm{P}}(t_0)\right], \tag{25}$$

from which the short-time evolution of the mean photon number is obtained as

$$\bar{n}(t_0+\tau) =< \Psi_0|\hat{n}(t_0+\tau)|\Psi_0 >= \bar{n}(t_0)\left\{1+2\kappa^2(N_2-N_1)\pi\tau\right\}+2\kappa^2 N_2\pi\tau. \tag{26}$$

In deriving Eq. (26), higher-order interaction terms are neglected, because the short-time evolution is considered here. This short-time evolution is translated to the short-length evolution along the medium length as

$$\bar{n}(z_0+\Delta z) = \bar{n}(z_0)\{1+g(N_2-N_1)\Delta z\}+gN_2\Delta z, \tag{27}$$

from which the following spatial differential equation is obtained:

$$\frac{d\bar{n}}{dz} = g(N_2-N_1)\bar{n}+gN_2. \tag{28}$$

This equation is equivalent to the photon-number rate equation given in Eq. (1). Therefore, similar to Eq. (1), the output of the mean photon number is calculated as

$$\bar{n}(L) = G\bar{n}(0)+n_{\text{sp}}(G-1). \tag{29}$$

The first and second terms represent amplified signal photons and ASE photons, respectively. It is noted that ASE photons appear at the output even though there is no such light in the mean amplitude as shown in Eq. (24).

### 4.4. Amplitude fluctuation

We next discuss amplitude fluctuations or noise. The light amplitude has two quadratures, i.e., the real and imaginary components. The operators representing each component are

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$\widehat{x}_1 = \left(\widehat{a} + \widehat{a}^\dagger\right)/2$ and $\widehat{x}_2 = \left(\widehat{a} - \widehat{a}^\dagger\right)/2i$, respectively, and their fluctuations are evaluated by $\sigma_{1,2}^2 = <\Psi_0|\widehat{x}_{1,2}^2|\Psi_0> - <\Psi_0|\widehat{x}_{1,2}|\Psi_0>^2$.

From Eq. (14), the short-time evolution of the mean square of the real component is expressed as

$$
\begin{aligned}
<\Psi_0|\widehat{x}_1^{\,2}(t_0+\tau)|\Psi_0> &= \frac{1}{4}<\Psi_0|\left\{\widehat{a}(t_0+\tau) + \widehat{a}^\dagger(t_0+\tau)\right\}^2|\Psi_0> \\
&= <\Psi|\widehat{x}_1^2(t_0)|\Psi> \left\{1 + 2\kappa^2(N_2 - N_1)\pi\tau\right\} + \frac{1}{2}\kappa^2(N_2 + N_1)\pi\tau,
\end{aligned} \tag{30}
$$

where $O(\tau^2)$ terms are neglected. On the other hand, the square of the average of the real component is expressed as

$$
\begin{aligned}
<\Psi_0|\widehat{x}_1(t_0+\tau)|\Psi_0>^2 &= \frac{1}{4}<\Psi_0|\left\{\widehat{a}(t_0+\tau) + \widehat{a}^\dagger(t_0+\tau)\right\}|\Psi_0>^2 \\
&= \frac{1}{4}\Big[<\Psi|\widehat{a}(t_0)|\Psi> \left\{1 + \kappa^2(N_2 - N_1)\pi\tau\right\} + <\Psi|\widehat{a}^\dagger(t_0)|\Psi> \\
&\left\{1 + \kappa^2(N_2 - N_1)\pi\tau\right\}\Big]^2 \approx <\Psi|\widehat{x}_1(t_0)|\Psi>^2\left\{1 + 2\kappa^2(N_2 - N_1)\pi\tau\right\}.
\end{aligned} \tag{31}
$$

From Eqs. (30) and (31), the short-time evolution of the variance of the real component is obtained as

$$
\begin{aligned}
\sigma_{x1}^2(t_0+\tau) &= <\Psi_0|\widehat{x}_1^{\,2}(t_0+\tau)|\Psi_0> - <\Psi_0|\widehat{x}_1(t_0+\tau)|\Psi_0>^2 \\
&= \left\{<\Psi|\widehat{x}_1^2(t_0)|\Psi> - <\Psi|\widehat{x}_1(t_0)|\Psi>^2\right\}\left\{1 + 2\kappa^2(N_2 - N_1)\pi\tau\right\} \\
&+ \frac{1}{2}\kappa^2(N_2 + N_1)\pi\tau = \sigma_{x1}^2(t_0)\left\{1 + 2\kappa^2(N_2 - N_1)\pi\tau\right\} \\
&+ \frac{1}{2}\kappa^2(N_2 + N_1)\pi\tau.
\end{aligned} \tag{32}
$$

This equation is translated to the short-length evolution as

$$
\sigma_{x1}^2(z_0 + \Delta z) = \sigma_{x1}^2(z_0)\{1 + g(N_2 - N_1)\Delta z\} + \frac{g}{4}(N_2 + N_1)\Delta z, \tag{33}
$$

from which the following differential equation is obtained:

$$
\frac{d\sigma_{x1}^2}{dz} = g(N_2 - N_1)\sigma_{x1}^2 + \frac{g}{4}(N_2 + N_1). \tag{34}
$$

From Eq. (34), the variance of the real component at the output is calculated as

$$\sigma_{x1}^2(L) = G\sigma_{x1}^2(0) + \frac{1}{4}\left(2n_{sp} - 1\right)(G - 1). \quad (35)$$

The variance of the imaginary component is similarly calculated as $\sigma_{x2}{}^2(L) = G\sigma_{x2}{}^2\ (0) + (1/4)(2n_{sp} - 1)(G - 1)$. The first term in Eq. (35) represents amplified fluctuations from the incident light, with a gain $G$ whose square root equals the amplitude gain [Eq. (24)], and the second term represents additional fluctuations that are superimposed onto the amplified fluctuation through the amplification process. This input and output relationship of amplitude fluctuations can be schematically illustrated in the complex amplitude space (constellation) as shown in **Figure 1**.

For a coherent incident state, i.e., whose amplitude variances is $\sigma_{x1}{}^2\ (0) = \sigma_{x2}{}^2\ (0) = 1/4$ [4], Eq. (35) is rewritten as

$$\sigma_{x1}^2(L) = \frac{1}{4}G + \frac{1}{4}\left(2n_{sp} - 1\right)(G - 1) = \frac{1}{4} + \frac{1}{2}n_{sp}(G - 1). \quad (36)$$

The first term 1/4 corresponds to the inherent quantum noise of a coherent state, and the second term represents amplitude fluctuations at the amplifier output in a classical picture. Recalling that the mean amplitude at the amplifier output is that amplified from the incident light with no addition mean field, as indicated in Eq. (24), Eq. (36) suggests that the amplifier output can be regarded as a summation of a clean signal light (i.e., coherent state), displaced from the initial mean amplitude position, and fluctuating light, whose mean value and variance are 0 and $n_{sp}(G - 1)/2$, respectively, in one quadrature. **Figure 2** illustrates this output condition in the complex amplitude space. Noting that the variance of the fluctuating light equals half of the spontaneous photon number indicated in the second term in Eq. (28), we can say that the above picture illustrated in **Figure 2** is equivalent to the classical picture of amplitude noise described in Section 2, where the ASE power is given by $\langle|E_{ASE}|^2\rangle = n_{sp}(G - 1)hf\Delta f$ and the variance of the real component of ASE light is given by $\langle\{\mathrm{Re}[E_{ASE}]\}^2\rangle - \langle\mathrm{Re}[E_{ASE}]\rangle = n_{sp}(G - 1)hf\Delta f/2$. Therefore, the classical picture introduced in Section 2 is confirmed by the quantum mechanical treatment presented here, except for the inherent quantum noise of 1/4. This noise 1/4 is sometimes called "vacuum fluctuation" or "zero-point fluctuation," that appears owing to quantum mechanics.

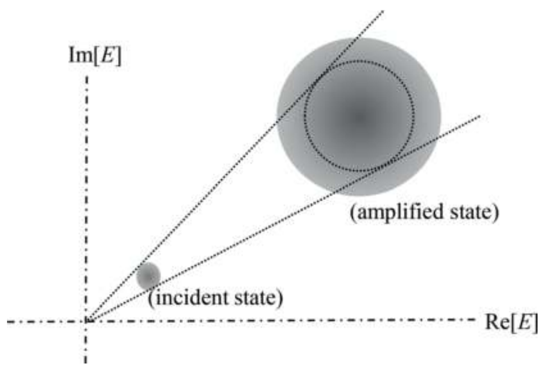


**Figure 1.** Constellation diagram of amplified light.

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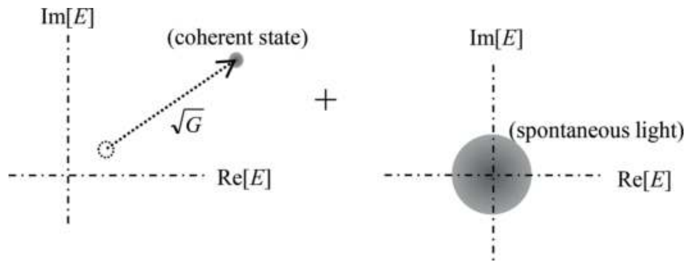


**Figure 2.** Decomposition of amplified light in the constellation diagram.

### 4.5. Photon-number fluctuation

We next discuss photon-number fluctuations. These fluctuations are evaluated employing the variance of the photon number as $\sigma_{\mathrm{n}}^2 =< \Psi_0|\hat{n}^2|\Psi_0 > - < \Psi_0|\hat{n}|\Psi_0>^2$. From Eq. (14), the short-time evolution of the mean square of the photon-number operator is calculated as

$$\begin{aligned} < \Psi_0|\hat{n}^{\,2}(t_0+\tau)|\Psi_0 > &=< \Psi_0|\left\{\hat{a}^\dagger(t_0+\tau)\hat{a}(t_0+\tau)\right\}^2|\Psi_0 > \\ &=< \Psi_0|\left[\left\{\hat{a}^\dagger(t_0)\left(1-\hat{\Pi}^\dagger(t_0)\right)+\hat{\mathrm{P}}^\dagger(t_0)\right\}\left\{\hat{a}(t_0)\left(1-\hat{\Pi}(t_0)\right)+\hat{\mathrm{P}}(t_0)\right\}\right]^2|\Psi_0 > \\ &=< \Psi|\hat{n}^2(t_0)|\Psi > \left\{1+4\kappa^2(N_2-N_1)\pi\tau\right\}+2\overline{n}(t_0)\kappa^2(3N_2+N_1)\pi\tau+2\kappa^2N_2\pi\tau, \end{aligned} \tag{37}$$

where higher-order interaction terms are neglected as before. This expression can be translated to the short-length evolution as

$$< \Psi_0|\hat{n}^2(z_0+\Delta z)|\Psi_0 >=< \Psi|\hat{n}^2(z_0)|\Psi > \{1+2g(N_2-N_1)\Delta z\}+g\overline{n}(z_0)(3N_2+N_1)\Delta z+gN_2\Delta z, \tag{38}$$

from which the following differential equation is obtained:

$$\frac{d < \hat{n}^2 >}{dz} = 2g(N_2-N_1) < \hat{n}^2 > +g(3N_2+N_1)\overline{n}+gN_2. \tag{39}$$

with $< \hat{n}^2 >=< \Psi|\hat{n}^2|\Psi >$. The first term represents an amplification process with a gain coefficient of $2g(N_2 - N_1)$, and the second and third terms represent the number of photons generated at a local position, which propagate and reach the medium end while being amplified by the first term. Then, the solution of Eq. (39) can be expressed as

$$< \hat{n}^2(L) >=< \hat{n}^2(0) > e^{2g(N_2-N_1)L} + \int_0^L \{g(3N_2+N_1)\overline{n}(z)+gN_2\}e^{2g(N_2-N_1)(L-z)}dz. \tag{40}$$

Here, $\overline{n}(z)$ is expressed from Eq. (29) as $\overline{n}(z)=\overline{n}(0)\exp[g(N_2-N_1)z]$, and then Eq. (40) is calculated as

$$< \hat{n}^2(L) >=< \hat{n}^2(0) > G^2 + n_{\mathrm{sp}}(G-1) + 4\overline{n}(0)G(G-1)n_{\mathrm{sp}} + 2n_{\mathrm{sp}}^2(G-1)^2 - \overline{n}(0)G(G-1). \tag{41}$$

From this equation and Eq. (29), the photon-number variance at the amplifier output is expressed as

$$\begin{aligned}\sigma_{\mathrm{n}}^2(L) >&=< \hat{n}^2(L) > -\{\overline{n}(L)\}^2 \\ &= 2\overline{n}(0)G(G-1)n_{\mathrm{sp}} + n_{\mathrm{sp}}^2(G-1)^2 + \overline{n}(0)G + n_{\mathrm{sp}}(G-1) \\ &\quad + G\left[< \hat{n}^2(0) > -\{\overline{n}(0)\}^2 - \overline{n}(0)\right].\end{aligned} \tag{42}$$

Recalling that the mean photon numbers of the amplified signal and the spontaneous emission are $\overline{n}(0)G$ and $n_{\mathrm{sp}}(G-1)$, respectively, each term in Eq. (42) can be interpreted as follows. The first term is equivalent to 2 × (signal light intensity) × (spontaneous light intensity), corresponding to the signal-spontaneous beat noise represented by the first term in Eq. (4). The second term is equivalent to (spontaneous light intensity)$^2$, corresponding to the spontaneous-spontaneous beat noise represented by the second and third terms in Eq. (4). The third and fourth terms denote the mean photon numbers of the amplified signal and spontaneous emission, respectively, corresponding to the inherent quantum noises of the amplified signal light and the spontaneous emission, respectively. In the fifth term, $< \hat{n}^2(0) > -\{\overline{n}(0)\}^2$ is the photon-number variance at the input and $\overline{n}(0)$ is that of a coherent state. Thus, their difference represents noise other than the inherent quantum noise, i.e., excess noise, and then the fifth term corresponds to the amplified excess noise.

The first and second terms in Eq. (42) correspond to the classical intensity noise represented by Eq. (4), as described above, supporting the classical treatment. In addition, the inherent quantum noises are included in Eq. (42), owing to the full quantum mechanical treatment, and the amplified excess noise is simultaneously included as well. Sometimes in the classical treatment, the inherent quantum noise is phenomenologically added as the shot noise arising at the electrical stage after direct detection [2]. In fact, however, it exists in the optical stage as derived above. Therefore, the inherent quantum noise is sometimes called "optical shot noise."

### 4.6. Noise figure

The noise figure, defined as the ratio of the signal-to-noise ratios (SNRs) at the input and output of an amplifier in terms of the light intensity or the photon number, is usually used as an indicator for the noise performance of an amplifier. Based on the above results, we describe the noise figure of population-inversion-based amplifiers in this subsection. The output SNR is obtained from Eqs. (29) and (42) as

$$(\mathrm{SNR})_{\mathrm{out}} = \frac{\{\overline{n}(0)G\}^2}{2\overline{n}(0)G(G-1)n_{\mathrm{sp}}}, \tag{43}$$

where only the signal power and the signal-spontaneous beat noise are taken into account, assuming that the amplified signal is sufficiently larger than the spontaneous emission. On the other hand, the input SNR is evaluated for a coherent state, according to the definition of the noise figure, which is $(\mathrm{SNR})_{\mathrm{in}} = \overline{n}(0)$. Therefore, the noise figure is expressed as

$$\mathrm{NF} = \frac{(\mathrm{SNR})_{\mathrm{in}}}{(\mathrm{SNR})_{\mathrm{out}}} = 2n_{\mathrm{sp}}\frac{G-1}{G}. \tag{44}$$

This expression equals the classical result given by Eq. (6).

The noise figure is proportional to the population inversion parameter $n_{sp} = N_2/(N_2 - N_1)$, as shown above. The minimum value of $n_{sp}$ is 1, which is achieved when $N_1 = 0$, i.e., the fully inverted condition where all atoms are in the upper states. Under this condition, NF = 3 dB for $G >> 1$. This is the quantum-limited noise figure of population-inversion-based optical amplifiers. Near-quantum-limited noise figure has been demonstrated experimentally in Erbium-doped fiber amplifiers [6, 7].

The fact that the noise performance is determined by the population inversion parameter can be intuitively understood as follows. The source of amplifier noise is spontaneous emission. A small amount of spontaneous emission suggests a good noise performance. However, spontaneous emission is roughly proportional to the signal gain (Eq. (29)), which is desired to be high as an amplifier. Thus, the amount of spontaneous emission normalized to the signal gain, (ASE power)/(signal gain), can be an indicator for the noise performance. The spontaneous emission rate is proportional to the number of atoms in the upper energy level $N_2$, i.e., (ASE power)$\propto N_2$, and the signal gain is determined by the balance between stimulated emission and absorption and thus is proportional to the difference between the numbers of atoms in the upper and lower states, roughly speaking, i.e., (signal gain)$\propto(N_2 - N_1)$. Subsequently, (ASE power)/(signal gain)$\propto N_2/(N_2 - N_1) = n_{sp}$, which suggests that the amplifier noise performance is determined by the population inversion parameter $n_{sp}$.

## 5. Optical parametric amplifiers

Whereas population-inversion–based amplifiers are widely used, there is another type of optical amplifiers, that is an optical parametric amplifier (OPA) based on optical nonlinearity [8]. When signal light is incident onto a nonlinear medium along with intense pump light, a signal and idler photons are created from one pump photon in case of second-order nonlinearity, satisfying the energy conservation of $\hbar\omega_s + \hbar\omega_i = \hbar\omega_p$ ($\omega_s$, $\omega_i$, and $\omega_p$ are the angular frequencies of the signal, idler, and pump lights, respectively), or they are created from two pump photons in case of third-order nonlinearity, satisfying $\hbar\omega_s + \hbar\omega_i = \hbar\omega_{p1} + \hbar\omega_{p2}$. Through this photon exchange phenomenon, the signal light is amplified. This signal amplification scheme also offers optical signal processing functions such as wavelength conversion and generation of phase-conjugated light [9]. This section describes quantum noise in OPAs [10].

### 5.1. Heisenberg equation

The Hamiltonian for parametric interaction between signal and idler via pump light(s) can be expressed as [11]

$$\widehat{H} = \hbar\omega_s\widehat{a}_s^\dagger\widehat{a}_s + \hbar\omega_i\widehat{a}_i^\dagger\widehat{a}_i + i\hbar\left(\chi\widehat{a}_s^\dagger\widehat{a}_i^\dagger - \chi^*\widehat{a}_s\widehat{a}_i\right). \tag{45}$$

The first and second terms are the Hamiltonians of signal and idler lights without interaction, respectively, where $\widehat{a}_s$ and $\widehat{a}_i$ are the field operators of signal and idler, respectively. The third term is the interaction Hamiltonian between signal, idler, and pump lights, which represents photon energy exchange such that signal and idler photons are created while pump photon(s) is annihilated and vice versa, with the coupling coefficient $\chi$. Since the pump light is so intense that its quantum properties do not matter here, the pump light is treated classically, whose amplitude $E_p$ is included in the coupling coefficient as $\chi \propto E_p$ or $E_{p1}E_{p2}$ for the second- or third-order nonlinear interaction, respectively.

From the Heisenberg equation with the above Hamiltonian, temporal differential equations for the field operators are obtained as

$$\frac{d\widehat{a}_s}{dt} = \frac{1}{i\hbar}\left[\widehat{a}_s, \widehat{H}\right] = -i\omega_s\widehat{a}_s + \chi\widehat{a}_i^\dagger, \tag{46a}$$

$$\frac{d\widehat{a}_i^\dagger}{dt} = \frac{1}{i\hbar}\left[\widehat{a}_i^\dagger, \widehat{H}\right] = i\omega_i\widehat{a}_i^\dagger + \chi^*\widehat{a}_s. \tag{46b}$$

These temporal differential equations can be translated to spatial ones as

$$\frac{d\widehat{a}_s}{dz} = -i\beta_s\widehat{a}_s + \frac{\chi}{(c/n)}\widehat{a}_i^\dagger, \tag{47a}$$

$$\frac{d\widehat{a}_i^\dagger}{dz} = i\beta_i\widehat{a}_i^\dagger + \frac{\chi^*}{(c/n)}\widehat{a}_s. \tag{47b}$$

where $\beta = n\ (\omega/c)$ is the propagation constant ($n$: the refractive index, $c$: the light velocity in the vacuum). The above equations can be simplified by the variable translation $\widehat{a}_{s,i}(z) \to \widehat{a}_{s,i}(z)$ $\exp\left(-i\beta_{s,i}z\right)$ as

$$\frac{d\widehat{a}_s}{dz} = \frac{\chi}{(c/n)}\widehat{a}_i^\dagger e^{i\left(\beta_s+\beta_i\right)z}, \tag{48a}$$

$$\frac{d\widehat{a}_i^\dagger}{dz} = \frac{\chi^*}{(c/n)}\widehat{a}_s e^{-i\left(\beta_s+\beta_i\right)z}. \tag{48b}$$

Here, we consider the propagation phase of the right-hand term in the above equations. The coefficient $\chi$ includes the pump light amplitude as $\chi \propto E_p$ or $E_{p1}E_{p2}$, and the pump amplitude can be expressed as $E_p = E_p\ (0)\exp(-i\beta_p z)$ under no pump-depletion condition ($\beta_p$ is the propagation constant of the pump light). Subsequently, $\chi \propto E_p\ (0)\exp(-i\beta_p z)$ or $E_{p1}(0)E_{p2}(0)$ $\exp[-i(\beta_{p1} + \beta_{p2})z]$. From these considerations, Eq. (48) can be rewritten as

$$\frac{d\widehat{a}_s}{dz} = \kappa\widehat{a}_i^\dagger e^{i\Delta\beta z}, \tag{49a}$$

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$$\frac{d\hat{a}_{\mathrm{i}}^{\dagger}}{dz} = \kappa^{*}\hat{a}_{\mathrm{s}}e^{-i\Delta\beta z}. \tag{49b}$$

where $\kappa$ is the coupling coefficient in the spatial domain, excluding $\exp(-i\beta_{\mathrm{p}}z)$ or $\exp[-i(\beta_{\mathrm{p1}} + \beta_{\mathrm{p2}})z]$, and $\Delta\beta \equiv \beta_{\mathrm{s}} + \beta_{\mathrm{i}} - \beta_{\mathrm{p}}$ or $\beta_{\mathrm{s}} + \beta_{\mathrm{i}} - \beta_{\mathrm{p1}} - \beta_{\mathrm{p2}}$. This parameter $\Delta\beta$ is called "phase mismatch," and determines the signal gain of an OPA as shown later. As for $\kappa$, its absolute value is $|\kappa| = d\gamma|E_{\mathrm{p1}}|\,|E_{\mathrm{p2}}|$ when an optical fiber is used as a nonlinear medium, where $\gamma$ is the nonlinear coefficient, and $d$ is the degeneracy factor that takes 2 and 1 for $f_{\mathrm{p1}} \neq f_{\mathrm{p2}}$ and $f_{\mathrm{p1}} = f_{\mathrm{p2}}$, respectively. Regarding the phase of $\kappa$, it is determined by the incident phase(s) of the pump light(s).

From Eq. (49), the signal field operator at the output is calculated as

$$\hat{a}_{\mathrm{s}}(L) = \left\{\cosh(gL) - i\left(\Delta\beta/2g\right)\sinh(gL)\right\}\hat{a}_{\mathrm{s}}(0) + e^{i\varphi}\sqrt{1+\left(\Delta\beta/2g\right)^{2}}\sinh(gL)\hat{a}_{\mathrm{i}}^{\dagger}(0), \tag{50a}$$

where $g \equiv \{|\kappa|^2 - (\Delta\beta/2)^2\}^{1/2}$, $L$ is the medium length, and $\varphi \equiv \arg(\kappa)$. Eq. (50a) includes the field operators of signal and idler lights, i.e., the signal and the idler are treated separately. For particular frequency conditions such as $2\omega_{\mathrm{s}} = \omega_{\mathrm{p}}$ for the second-order nonlinearity or $2\omega_{\mathrm{s}} = \omega_{\mathrm{p1}} + \omega_{\mathrm{p2}}$ for the third-order nonlinearity, the idler frequency equals to the signal frequency, $\omega_{\mathrm{i}} = \omega_{\mathrm{s}}$, and the signal and the idler are degenerate. Under such conditions, Eq. (50a) is rewritten as

$$\hat{a}_{\mathrm{s}}(L) = \left\{\cosh(gL) - i\left(\Delta\beta/2g\right)\sinh(gL)\right\}\hat{a}_{\mathrm{s}}(0) + e^{i\varphi}\sqrt{1+\left(\Delta\beta/2g\right)^{2}}\sinh(gL)\hat{a}_{\mathrm{s}}^{\dagger}(0), \tag{50b}$$

As shown later, degenerate and nondegenerate OPAs have definitely different characteristics. For simplifying mathematical expressions, hereafter, we rewrite Eq. (50) as

$$\hat{a}_{\mathrm{s}}(L) = \left\{Ae^{i(\varphi-\phi)/2}\hat{a}_{\mathrm{s}}(0) + Be^{-i(\varphi-\phi)/2}\hat{a}_{\mathrm{i,s}}^{\dagger}(0)\right\}e^{i(\varphi+\phi)/2}, \tag{51}$$

with

$$A = \sqrt{\cosh^{2}(gL) + \left(\Delta\beta/2g\right)^{2}\sinh^{2}(gL)}, \tag{52a}$$

$$\phi = \arctan\left[-\frac{\left(\Delta\beta/2g\right)\sinh(gL)}{\cosh(gL)}\right], \tag{52b}$$

$$B = \sqrt{1+\left(\Delta\beta/2g\right)^{2}}\sinh(gL). \tag{52c}$$

The mean values of the physical quantities after amplification can be evaluated using Eq. (51). In the evaluation, we need the initial state in addition. Here, we assume that only signal light is incident to an OPA, and express the initial state as $|\Psi_0\rangle = |\Psi\rangle_{\mathrm{s}}\otimes|0\rangle_{\mathrm{i}}$, where $|\Psi\rangle_{\mathrm{s}}$ and $|0\rangle_{\mathrm{i}}$ denote the signal and idler states, respectively. When the initial state is a coherent state as $|\Psi\rangle_{\mathrm{s}} = |\alpha\rangle$, we have $\hat{a}_{\mathrm{s}}(0)|\Psi\rangle = \sqrt{\bar{n}_{\mathrm{s}}(0)}e^{i\theta}|\Psi\rangle$, where $\bar{n}_{\mathrm{s}}(0)$ and $\theta$ are the mean photon number and the phase of the incident signal light, respectively. On the other hand, $\hat{a}_{\mathrm{i}}(0)|0\rangle_{\mathrm{i}} = 0$.

### 5.2. Mean amplitude, photon number, and signal gain

The mean amplitude and photon number at the output are evaluated by $\bar{a}_s(L) =< \Psi_0|\hat{a}_s(L)|\Psi_0 >$ and $\bar{n}_s(L) =< \Psi_0|\hat{a}_s^\dagger(L)\hat{a}_s(L)|\Psi_0 >$, respectively. For nondegenerate OPA, these are calculated from Eq. (51) as

$$\bar{a}_s(L) = \bar{a}_s(0)Ae^{i\varphi}, \tag{53}$$

$$\bar{n}_s(L) = \bar{n}_s(0)A^2 + B^2. \tag{54}$$

Eq. (53) indicates that the signal field is simply amplified while preserving the phase state, with no additional field on average. On the other hand, Eq. (54) shows that the output photons consist of two components. The first term is proportional to the incident photon number, which corresponds to the amplified signal photons with a gain of

$$A^2 = \cosh^2(gL) + \left(\Delta\beta/2g\right)^2\sinh^2(gL) \equiv G. \tag{55}$$

It is noted in this expression that parameter $g = \{|\kappa|^2 - (\Delta\beta/2)^2\}^{1/2}$ is equivalent to the gain coefficient. When $\Delta\beta = 0$, $g$ is maximum and the signal gain is maximum. Therefore, it is important to satisfy the condition $\Delta\beta = 0$, that is called the "phase matching condition," in implementing an OPA [9]. The second term in Eq. (54) is independent on the signal input, and represents spontaneously emitted photons. The above results, i.e., the spontaneous light does not appear in the mean amplitude while it does in the photon number, suggest that the amplitude of the spontaneous emission is completely random. However, we do not know how random it is at this stage. The photon number of the spontaneous emission is expressed from the second term in Eqs. (54) and (52c) as

$$B^2 = \left\{1 + \left(\Delta\beta/2g\right)^2\right\}\sinh^2(gL) = G - 1, \tag{56}$$

where Eq. (55) is applied. This expression is equivalent to the spontaneous photon number in population-inversion-based amplifiers indicated in Eq. (29) with $n_{sp} = 1$. This correspondence suggests that nondegenerate OPAs can offer the ideal noise performance achievable in EDFAs, which is shown later.

Regarding degenerate OPA, on the other hand, its mean output amplitude is calculated as

$$\bar{a}_s(L) = \left\{Ae^{i\{\theta_0+(\varphi-\phi)/2\}} + Be^{-i\{\theta_0+(\varphi-\phi)/2\}}\right\}|\bar{a}_s(0)|e^{i(\varphi+\phi)/2}, \tag{57}$$

where $\theta_0$ is the phase of the incident signal light. The mean output amplitude does not have a simple form as in nondegenerate OPA (Eq. (53)). Under the condition where $\Delta\beta = 0$ and the gain coefficient $g$ is so large as $\cosh(gL) \approx \sinh(gL) \approx e^{gL}/2$, Eq. (57) is approximated as

$$\bar{a}_s(L) = \cos(\Delta)e^{gL}|\bar{a}_s(0)|e^{i(\varphi+\phi)/2}, \tag{58}$$

where $\Delta \equiv \theta_0 + (\varphi - \phi)/2$ is introduced. This expression indicates that the phase state of the incident signal light is not transferred to the output.

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The mean photon number in degenerate OPA is calculated from Eq. (51) as

$$\overline{n}_s(L) = \left(A^2 + B^2\right)\overline{n}_s(0) + AB\left\{< \Psi|\left[\{\hat{a}_s(0)\}^2 e^{i(\varphi-\phi)/2} + \left\{\hat{a}_s^\dagger(0)\right\}^2 e^{-i(\varphi-\phi)/2}\right]|\Psi >\right\} + B^2. \quad (59)$$

Unfortunately, this equation cannot be further developed, because we cannot readily calculate $< \Psi|\{\hat{a}_s(0)\}^2|\Psi >$ and $< \Psi|\left\{\hat{a}_s^\dagger(0)\right\}^2|\Psi >$ for an arbitrary $|\Psi>$. However, for a coherent incident state, these quantities can be evaluated using $\hat{a}_s(0)|\Psi >= \sqrt{\overline{n}_s(0)}e^{i\theta}|\Psi >$, and then Eq. (59) is developed as

$$\overline{n}_s(L) = \overline{n}_s(0)\left\{A^2 + B^2 + 2AB\cos(2\Delta)\right\} + B^2. \quad (60)$$

In this expression, the first term represents amplified signal photons, and the second term represents spontaneous emission whose mean amplitude is zero as indicated in Eq. (57).

From the first term in Eq. (60), the signal gain is expressed as

$$G = A^2 + B^2 + 2AB\cos(2\Delta), \quad (61)$$

which is dependent on the relative phase $\Delta = \theta_0 + (\varphi - \phi)/2$. Hence, degenerate OPA is called "phase-sensitive amplifier (PSA)." The maximum gain is obtained when $\Delta = 0$ as

$$G(\Delta = 0) = (A + B)^2. \quad (62)$$

### 5.3. Amplitude fluctuation

Next, we evaluate the amplitude noise in OPAs. For the evaluation, the light amplitude is decomposed into two quadratures and the variance of each quadrature is calculated, as in Section 4.3. In case of OPAs, the output field operator is phase-shifted by $(\varphi + \phi)/2$, as indicated by Eq. (51). Accordingly, we introduce a phase-shifted field operator defined as $\hat{b} \equiv \hat{a}_s e^{-i(\phi+\varphi)/2}$, and evaluate the variances of the real and imaginary components of $\hat{b}$, i.e., $\hat{x}_1 = \left(\hat{b} + \hat{b}^\dagger\right)/2$ and $\hat{x}_2 = \left(\hat{b} - \hat{b}^\dagger\right)/2i$, respectively, as $\sigma_{x1(2)}^2 = <\Psi_0|\hat{x}_{1,2}^2|\Psi_0> - <\Psi_0|\hat{x}_{1,2}|\Psi_0>^2$.

The calculation result for nondegenerate OPA is expressed as

$$\sigma_{x1(2)}^2(L) = A^2\sigma_{x1(2)}^2(0) + \frac{1}{4}B^2 = A^2\sigma_{x1(2)}^2(0) + \frac{1}{4}(G - 1), \quad (63)$$

where Eqs. (55) and (56) are applied. The first term represents noise amplified from the incident light, and the second term represents additional noise superimposed via OPA. Note that Eq. (63) is equivalent to the amplitude variance of population-inversion-based amplifiers shown by Eq. (35) with $n_{sp} = 1$, suggesting that the ideal noise performance achievable in EDFAs can be obtained in OPA. Similar to Eq. (35), Eq. (63) is rewritten for a coherent incident state as

$$\sigma^2_{x1(2)}(L) = \frac{1}{4} + \frac{1}{2}(G-1). \tag{64}$$

The first term is the inherent quantum noise of a coherent state, and the second term represents noise superimposed via OPA in a classical picture. The sum of the second terms of the two quadratures equals the photon number of spontaneous emission light indicated in Eq. (56). This consideration supports the classical noise treatment, described in Section 2, where spontaneous emission with random phase is superimposed onto signal light at the amplifier output.

For degenerate OPA, on the other hand, the amplitude variances are calculated as

$$\sigma^2_{x1}(L) = (A+B)^2\sigma^2_{x1}(0) = G_0\sigma^2_{x1}(0), \tag{65a}$$

$$\sigma^2_{x2}(L) = (A-B)^2\sigma^2_{x2}(0), \tag{65b}$$

where $G_0 \equiv G(\Delta = 0)$ is the phase-synchronized gain introduced in Eq. (62). The results show that the output variances are unequal in the two quadratures, such that ${\sigma_{x1}}^2$ is enhanced while ${\sigma_{x2}}^2$ is depressed. Since $\hat{x}_1$ and $\hat{x}_2$ are the real and imaginary parts of the phase-shifted operator, respectively, this result suggests that amplitude noise along the axis of a phase of $(\varphi + \phi)/2$ is enhanced and that along the orthogonal axis is depressed, in the complex amplitude space. It is noted in Eq. (65a) that it only has fluctuations amplified from incident light with no additional fluctuation, unlike population-inversion-based amplifiers (Eq. (35)) and nondegenerate OPA (Eq. (63)). This result suggests that even the noise-enhanced quadrature $\hat{x}_1$ is expected to have lower noise than these amplifiers. Regarding the imaginary part, Eq. (65b) for high-gain conditions, where $\cosh(gL) \approx \sinh(gL) \approx e^{gL}/2$ and then $A \approx B$, is rewritten as ${\sigma_{x1}}^2 \approx 0$. This consideration indicates that the amplitude noise along the phase axis orthogonal to the signal output phase approaches zero as the signal gain increases. The above-mentioned characteristics of amplitude fluctuation in degenerate OPA can be illustrated in the complex amplitude space as shown in **Figure 3**.

### 5.4. Photon-number fluctuation and noise figure

Next, we discusses photon-number fluctuations in OPAs, which are evaluated through the photon-number variance as ${\sigma_n}^2 = <\Psi_0|\hat{n}^2|\Psi_0> - <\Psi_0|\hat{n}|\Psi_0>^2$. Using Eq. (51), the average of the square of the photon-number operator for nondegenerate OPA is calculated as

$$<\Psi_0|\hat{n}^2_s(L)|\Psi_0> = <\Psi_0|\left\{\hat{a}^\dagger_s(L)\hat{a}_s(L)\right\}^2|\Psi_0> = <\Psi_0|\hat{n}_s(L)|\Psi_0>^2 + \left(A^2+B^2\right)A^2\overline{n}_s(0) + (AB)^2. \tag{66}$$

Subsequently, the photon-number variance is obtained as

$$\begin{aligned}\sigma^2_n(L) &= \left(A^2+B^2\right)A^2\overline{n}_s(0) + (AB)^2 = (2G-1)G\overline{n}_s(0) + G(G-1)\\ &= 2(G-1)G\overline{n}_s(0) + (G-1)^2 + G\overline{n}_s(0) + G(G-1),\end{aligned} \tag{67}$$

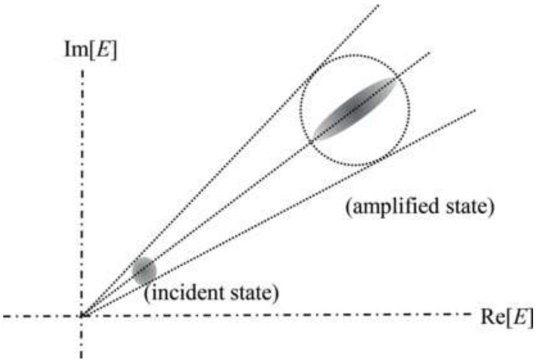


**Figure 3.** Constellation diagram of amplified light in a phase-synchronized degenerate OPA.

where Eqs. (55) and (56) are applied. Recalling that the mean photon number of the amplified signal is $G\overline{n}_s(0)$ and that of spontaneous emission light is $(G-1)$, as indicated by Eqs. (54)–(56), each term in Eq. (68) can be interpreted as follows. The first term is equivalent to 2 × (signal light intensity) × (spontaneous light intensity), which corresponds to the signal-spontaneous beat noise. The second term is equivalent to (spontaneous light intensity)$^2$, which corresponds to the spontaneous-spontaneous beat noise. The third and fourth terms are equal to the mean photon numbers of the amplified signal and of spontaneous emission, respectively, and correspond to the optical shot noises of the amplified signal light and the spontaneous emission, respectively.

Next, we consider degenerate OPA. As indicated by Eq. (59), properties of the photon number in degenerate OPA are hard to evaluate for an arbitrary initial state. Thus, we assume a coherent incident state here. From Eq. (51), the average of the square of the photon-number operator for the initial state $|\Psi_0\rangle = |\alpha\rangle_s \otimes |0\rangle_i$ is calculated as

$$\begin{aligned} <\Psi_0|\hat{n}_s^2(L)|\Psi_0> &= [A^2+B^2+2AB\cos(2\Delta)\overline{n}_s(0)+B^2]^2 \\ &\quad +\left\{\left(A^2+B^2\right)^2+4(AB)^2+4\left(A^2+B^2\right)AB\cos(2\Delta)\right\}\overline{n}_s(0)+2(AB)^2 \\ &= <\Psi_0|\hat{n}_s(L)|\Psi_0>^2+\left\{\left(A^2+B^2\right)^2+4(AB)^2+4\left(A^2+B^2\right)AB\cos(2\Delta)\right\}\overline{n}_s(0)+2(AB)^2, \end{aligned} \quad (68)$$

Subsequently, the photon-number variance at the output is

$$\begin{aligned} \sigma_n^2(L) &= \left\{\left(A^2+B^2\right)^2+4(AB)^2+4\left(A^2+B^2\right)AB\cos(2\Delta)\right\}\overline{n}_s(0)+2(AB)^2 \\ &= \left\{G^2+4(AB)^2\sin^2(2\Delta)\right\}\overline{n}_s(0)+2(AB)^2, \end{aligned} \quad (69)$$

where Eq. (61) is applied. This expression cannot be decomposed and interpreted as that of nondegenerate OPA indicated by Eq. (68), which could be because the amplitude distribution is not simply isotropic in two quadratures, unlike nondegenerate OPA.

The noise figure can be evaluated from the results obtained above. For nondegenerate OPA, it is obtained as

$$\mathrm{NF} = \frac{(\mathrm{SNR})_{\mathrm{in}}}{(\mathrm{SNR})_{\mathrm{out}}} = \overline{n}_{\mathrm{s}}(0)\frac{2(G-1)G\overline{n}_{\mathrm{s}}(0)}{\{G\overline{n}_{\mathrm{s}}(0)\}^2} = 2\frac{G-1}{G}. \tag{70}$$

where only the signal-spontaneous beat noise is considered for the output SNR, according to the definition on the noise figure. This noise figure equals that of ideal population-inversion–based amplifiers indicated by Eq. (44) with $n_{\mathrm{sp}} = 1$. For the degenerate case, on the other hand, it is expressed as

$$\mathrm{NF} = 1 + 4\frac{(AB)^2}{G}\sin^2(2\Delta). \tag{71}$$

For $\Delta = 0$, NF = 1 (0 dB), suggesting no SNR degradation in phase-synchronized degenerate OPA. In fact, a noise figure of less than 3 dB in a phase-sensitive amplifier has been experimentally demonstrated [12, 13].

## 6. Conclusion

This chapter describes quantum noise of optical amplifiers. Full quantum mechanical treatment based on the Heisenberg equation for physical quantity operators was presented, by which quantum properties of optical amplifiers were derived from first principles. The obtained results are consistent with a conventional classical treatment, except for the inherent quantum noise or the zero-point fluctuation, providing the theoretical base to the conventional phenomenological treatment.

## Author details

Kyo Inoue

Address all correspondence to: kyo@comm.eng.osaka-ua.c.jp

Osaka University, Osaka, Japan

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