Chapter 7

Development of Recurrent Method with Rotation for Combined Adjustment of Terrestrial Geodetic and GNSS Networks in National Spatial Reference System

Ha Minh Hoa

Additional information is available at the end of the chapter

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Abstract

A construction of national spatial reference systems (NSRS) is promoted in many countries due to modern achievements of Global Navigation Satellite System (GNSS) methods and results of building of high accurate geoid/quasi-geoid models at centimeter level of accuracy. One of the most popular methods used for the construction of the NSRS is related to Helmert block adjustment method, by which we ought to solve techno-scientific task of a separate adjustment of GNSS network in International Terrestrial Reference Frame (ITRF) and next combination of a results of adjustment of the terrestrial geodetic and GNSS networks in the NSRS. In this chapter, we carry out a research on the usage of a recurrent adjustment method with Givens rotation for solving the abovementioned task on an account of its advantages of being effective for application of a technique of sparse matrix, outlier detection and very simple for solving the subsystem of observation equations, created based on the transformation of the results of the separate adjustment of the GNSS network from the ITRF into the NSRS. The experiment results of solving the abovementioned task for the GPS network in the North Vietnam had shown that the horizontal and vertical position accuracy of the GPS points in VN2000–3D had reached the few centimeter level.

Keywords: method of recurrent adjustment, combined adjustment of terrestrial geodetic and GNSS networks, recurrent adjustment method with rotation, method of Givens rotation, national spatial reference system

1. Introduction

In the past, in different countries, national horizontal and vertical reference systems had been constructed independently from each other; in addition, horizontal control points almost did not coincide with vertical control points. A national first- and second-order astro-geodetic network constructed by traditional geodetic methods did not allow to obtain horizontal positioning accuracy of the horizontal control points at the centimeter level. Because of an accumulation of measurement errors in the national horizontal control network, a coordinate transmission from one origin point led to the more horizontal positioning error of the distant horizontal control points. For example, the horizontal position accuracy in NAD83 (1986) reached the level of 1 m [4, 34]. Such analogical situation had also been happened to the vertical control network. For the national first- and second-order astro-geodetic networks of the former Soviet Union in SK95, the maximal RMS of horizontal position of horizontal control points reached the level of 1.5 m [9].

Nowadays, traditional geodetic methods cannot satisfy the accuracy requirements of the national horizontal and vertical reference systems at the centimeter level according to modern techno-scientific achievements. The abovementioned accuracy requirements only will be satisfied by the construction of the NSRS based on modern achievements of the GNSS methods, the construction of the highly accurate national geoid/quasi-geoid model and the geopotential vertical datum.

Present-day worldwide and rapid development of GNSS methods, especially the construction of Continuously Operating Reference Station (CORS) networks of GNSS base stations and mathematical processing of GNSS data in the ITRF with usage of International GNSS Service (IGS) products, and construction of national hybrid geoid/quasi-geoid models with an accuracy at the level of few centimeters had created favorable conditions for building of the NSRS in many countries, for example, ETRS89/DREF91/2016 (Germany), GDA2020 (Australia) [14], NSRS2022 (USA CONUS, Canada, Caribbean Islands. Hawaii and Greenland) [35], and so on.

In case of processing the GNSS data in the ITRF, highly accurate spatial coordinates of geodetic points will be converted from the ITRF to the NSRS by the seven parameter Bursa-Wolf formula. Next, we symbolize $\mathbf{X_0}, \mathbf{Y_0}, \mathbf{Z_0}, \boldsymbol{\varepsilon_X}, \boldsymbol{\varepsilon_Y}, \boldsymbol{\varepsilon_Z}, \boldsymbol{\Delta m}$ as the seven coordinate transformation parameters from the ITRF to the NSRS by Bursa-Wolf formula, where $\mathbf{X_0}, \mathbf{Y_0}, \mathbf{Z_0}$ are the spatial coordinates of the origin of the ITRF with respect to the origin of the NSRS, $\boldsymbol{\varepsilon_X}, \boldsymbol{\varepsilon_Y}, \boldsymbol{\varepsilon_Z}$ are Euler rotation angles of the coordinate axes of the ITRF with respect to the analogical coordinate axes of the NSRS, $\boldsymbol{\Delta m}$ is a scale factor change.

For geodetic purposes, the NSRS contains an ellipsoidal surface used as the reference surface for the determination of an ellipsoidal coordinate system and a national plane coordinate system. A Geoid/quasigeoid surface is used for the reference surface of the national vertical reference system. In addition, the national geoid/quasigeoid model creates relationship of the geoid/quasi-geoid surface to the ellipsoidal surface and satisfies the connection of the spatial coordinates of geodetic points with the national vertical reference system.

In practice of the construction of GNSS network by the static relative positioning technique, the components $\boldsymbol{\Delta X}, \boldsymbol{\Delta Y}, \boldsymbol{\Delta Z}$ of baseline vector between two GNSS points obtained from the processing of GNSS observations have been used as measured values in the GNSS network. Using IGS products for processing GNSS observations in the ITRF, the components

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ΔX, ΔY, ΔZ of the baseline vectors have very high accuracy and have been used for the adjustment of the GNSS network. In [19], formulas for apriori assessment of relative horizontal position accuracy M_{xy} between two GNSS points and accuracy of ellipsoidal height $m_{\overline{H}}$ had been proposed in the following forms:

$$M_{xy} = \pm \frac{1}{\sqrt{2}}.\left(\frac{M_S}{1\ \ cm}\right).10^{-9}.b, \tag{1}$$

$$m_{\overline{H}} = \pm \frac{M_S}{\sqrt{3}}.secB, \tag{2}$$

where b - is the distance between two GNSS points in units of km; B - is the geodetic latitude of GNSS point; M_S (in units of cm) – accuracy of IGS precise ephemerides at the level of 2.5 cm (or 5 cm).

From formulas (1) and (2), we see that for the GNSS network, constructed by the static relative positioning technique, using IGS products for processing of GNSS observations in the ITRF enables very high relative horizontal position accuracy between two any GNSS points and the very high accuracy of ellipsoidal heights. The highly accurate GNSS network can be used for maintenance and improvement of the accuracy of the national horizontal and vertical reference systems. The construction of the NSRS will satisfy the abovementioned demands. For the construction of the NSRS, we can solve for the three of the following main techno-scientific tasks:

- Construction of the passive GNSS network, covering whole national territory;
- Construction of the national geoid/quasigeoid model with the accuracy at centimeter level;
- Combined adjustment of the terrestrial geodetic and passive GNSS networks in the NSRS.

With the purpose of the maintenance and the improvement of the accuracy of the national horizontal and vertical reference systems, apart from some of the CORS stations, the passive GNSS network still consists of horizontal and vertical control points which are called as ground control points and have been selected by the following criteria [1, 7]:

- Their location must satisfy requirements of a good satellite geometry and a sky visibility.
- Quick and easy access to them.
- Selected points may be located on geologically stable positions.

The passive GNSS points may have a 20–100 km density [1, 2, 6–8]. The passive GNSS networks have been built in many countries, for example High Accuracy Reference Network (USA) [33], Passive Control Network (Canada) [3, 38], Auscope GNSS Network (Australia), and so on.

Highly accurate ellipsoidal heights at the vertical control benchmarks which are derived from the processing of co-located GNSS observations in the ITRF, especially for the countries at the low and mid-latitudes, enable determining highly accurate geoid/quasi-geoid heights at those benchmarks. Those are very important data source for the determination of GNSS-leveling geoid/quasigeoid heights used for the improvement of accuracy of the national gravimetric geoid/quasi-geoid models.

Over the last decade, countries in Europe, South America, Canada, the United States of America, and so on had developed the geoid-based vertical reference systems (geopotential datum) [36, 37, 40]. An initial surface of the geopotential datum is the geoid surface with the geopotential $\mathbf{W_0}$. With usage of the geopotential datum, we have determined geopotentials of the vertical control benchmarks that will be used for the construction of the geopotential field model on the national territory or in a region. In addition to this, highly accurate ellipsoidal heights determined by the GNSS methods at the vertical control benchmarks allow calculating anomalous geopotentials of those control benchmarks which are additional data source for making more precision of spherical harmonic coefficients of the Earth Gravitational Model [21].

In [25], it is shown that in the NSRS, the relative accuracy of spatial coordinates may reach the level of 10^{-9}. Based on this criterion, in [20], it had been proved that the accuracy of the national geoid/quasi-geoid model can be improved to a level of higher than ±4 cm. At present, the national geoid/quasigeoid models in many countries, for example, AUSGeoid09 (Australia), USGG2012 (USA), CGG2013 (Canada), OSGM15 (UK), GCG2016 (Germany), and so on, have the accuracy higher than the abovementioned limitation, which guarantees to obtain orthometric/normal heights of points of interest with accuracy at the centimeter level based on the highly accurate national geoid/quasi-geoid model and results of GNSS data processing in the ITRF.

With the purpose of the maintenance and the improvement of accuracy of the national horizontal and vertical reference systems, in this chapter, we research on methods used for combined adjustment of terrestrial geodetic and passive GNSS networks, especially on a recurrent method with rotation for a combined adjustment of terrestrial geodetic and GNSS networks in the NSRS.

Although we use the terminology "combined adjustment of terrestrial geodetic and GNSS networks in the NSRS," the terrestrial geodetic network comprising horizontal and vertical control networks had been adjusted previously. Therefore, in this chapter, we understand this terminology as "combination of the results of separate adjustment of terrestrial geodetic and GNSS networks in the NSRS."

2. Methodology

2.1. Methods for combined adjustment of terrestrial geodetic and passive GNSS networks

A terrestrial geodetic network contains the national horizontal and vertical control networks that had been adjusted separately in the national horizontal and vertical reference systems. For the ground control points selected from the national horizontal and vertical control points and

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used for the construction of the passive GNSS network, their national ellipsoidal coordinates play very important role in solving the task of the combined adjustment of the terrestrial geodetic and the passive GNSS networks. The accuracy improvement of the national ellipsoidal coordinates (or corresponding spatial coordinates) of the abovementioned ground control points in the NSRS is the purpose of solving of the abovementioned task.

In common case, it is assumed that the national reference ellipsoid and the global reference ellipsoid are different form each other. For the national horizontal control points from results of processing of colocated GNSS observations in the ITRF according to the global reference ellipsoid, we will create relationship between the global geodetic latitudes, longitudes and the national geodetic latitudes, longitudes of these points by the Molodensky formula. This allows to obtain the national geodetic latitude, longitude of the vertical control benchmarks on which GNSS observations had been performed.

The orthometric/normal height of the national horizontal control points can be obtained by precise spirit (geometric) leveling or using a national geopotential field model with determined geopotential $\mathbf{W_0}$ of the national geoid. The first national geopotential field model in Vietnam had been declared in [23].

Such national ellipsoidal heights of the ground control points fully can be derived based on the highly accurate national geoid/quasigeoid model and the GNSS method. By such ways, we will obtain the national ellipsoidal coordinates of the ground control points, which will then be used for the calculation of approximate spatial coordinates $\mathbf{X}, \ \mathbf{Y}, \ \mathbf{Z}$ of these points in the NSRS.

In geodetic practice have been created two different directions related to development of methods for the combined adjustment of the terrestrial geodetic and GNSS networks. In the first direction, the components $\mathbf{\Delta X}, \ \mathbf{\Delta Y}, \ \mathbf{\Delta Z}$ of baseline vectors in the GNSS network have been used as pseudo-observations for the combined adjustment with different terrestrial observations on the national reference ellipsoid and for them in observation equations unknown parameters are ellipsoidal coordinate corrections and coordinate transformation parameters $\boldsymbol{\varepsilon_X}, \ \boldsymbol{\varepsilon_Y}, \ \boldsymbol{\varepsilon_Z}, \ \mathbf{\Delta m}$ [26]. In case the seven coordinate transformation parameters by Bursa-Wolf formula are known, the components $\Delta X, \ \Delta Y, \ \Delta Z$ of baseline vectors will be transformed from the ITRF to the NSRS. After that, those components $\mathbf{\Delta X}, \ \mathbf{\Delta Y}, \ \mathbf{\Delta Z}$ of baseline vectors can be transformed to $\mathbf{s}, \ \boldsymbol{\alpha}, \ \mathbf{\Delta h},$ where $\mathbf{s}, \ \boldsymbol{\alpha}$ - is length and azimuth of the geodesic; $\mathbf{\Delta h}$ - is the difference of ellipsoidal heights. The values $\mathbf{s}, \boldsymbol{\alpha}, \mathbf{\Delta h}$ will be used as pseudo-observations for the combined adjustment with various terrestrial observations on the national reference ellipsoid [26, 27].

The second direction is related to the development of methods for the combined adjustment of the terrestrial geodetic and GNSS networks based on the Helmert block method by principle: a separate adjustment of the terrestrial geodetic and GNSS networks and their next combination. The separate adjustment of the passive GNSS network will be performed with two following purposes:

- Outlier detection and their removal (if they exist) in the passive GNSS network.
- Determination of highly accurate spatial coordinates of the GNSS points in the ITRF.

We will continue the research of the second direction in the following contents of this chapter.

It is assumed that the passive GNSS network consists of NP points, in which np common points (np ≤ NP) are the ground control points. In addition, these points have the approximate spatial coordinates in the NSRS presented in the form of the national spatial coordinate vector:

$$\boldsymbol{\tau} = (\mathbf{X_1},\ \mathbf{Y_1},\ \mathbf{Z_1},...,\mathbf{X_{np}},\ \mathbf{Y_{np}},\ \mathbf{Z_{np}})^{\mathbf{T}} \tag{3}$$

with variance–covariance matrix $\mathbf{K_\tau^{kxk}} = \boldsymbol{\mu}_{\boldsymbol{\tau}}.\mathbf{Q}_{\boldsymbol{\tau}}$, where $k = 3.np$ - order of matrix; $\boldsymbol{\mu_\tau}$ - RMS of the unit weight determined apriori.

Without the loss in generality, we arrange ground control points in the first orders. After the separate adjustment of the passive GNSS network in the ITRF, we obtain the adjusted spatial coordinate vector of the NP GNSS points in following form:

$$\mathbf{S} = (\overline{\mathbf{X}}_1,\ \overline{\mathbf{Y}}_1,\ \overline{\mathbf{Z}}_1,...,\overline{\mathbf{X}}_{\mathbf{np}},\ \overline{\mathbf{Y}}_{\mathbf{np}},\ \overline{\mathbf{Z}}_{\mathbf{np}} \mid \overline{\mathbf{X}}_{\mathbf{np+1}},\ \overline{\mathbf{Y}}_{\mathbf{np+1}},\ \overline{\mathbf{Z}}_{\mathbf{np+1}},...,\overline{\mathbf{X}}_{\mathbf{NP}},\ \overline{\mathbf{Y}}_{\mathbf{NP}},\ \overline{\mathbf{Z}}_{\mathbf{NP}})^{\mathbf{T}} = (\mathbf{S_1} \mid \mathbf{S_2})^{\mathbf{T}}, \tag{4}$$

with variance-covariance matrix $\mathbf{K_S} = \boldsymbol{\mu}_{\mathbf{S}}^{\mathbf{2}}.\mathbf{R}_{\mathbf{S}}^{\mathbf{-1}}$, where μ_S is the RMS of the unit weight and $\mathbf{R_S}$ is the normal matrix of the order K obtained from the process of the separate adjustment of the passive GNSS network. In addition, the order K = 3.NP; $\mathbf{S_1}$ is a subvector of the spatial coordinates of the np ground control points in the ITRF; $\mathbf{S_2}$ is a subvector of the spatial coordinates of the remaining (NP – np) GNSS points in the ITRF.

In common case, for the GNSS points, the spatial coordinates $\overline{\mathbf{X}}, \overline{\mathbf{Y}}, \overline{\mathbf{Z}}$ in the ITRF are related to the spatial coordinates in the NSRS by Bursa-Wolf formula in the following form:

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\\ \mathbf{Z}\end{pmatrix} = \begin{pmatrix}\overline{\mathbf{X}}\\ \overline{\mathbf{Y}}\\ \overline{\mathbf{Z}}\end{pmatrix} + \begin{pmatrix}\mathbf{X_0}\\ \mathbf{Y_0}\\ \mathbf{Z_0}\end{pmatrix} + \begin{bmatrix}\boldsymbol{\Delta}\mathbf{m} & \boldsymbol{\varepsilon}_{\mathbf{Z}} & -\boldsymbol{\varepsilon}_{\mathbf{Y}}\\ -\boldsymbol{\varepsilon}_{\mathbf{Z}} & \boldsymbol{\Delta}\mathbf{m} & \boldsymbol{\varepsilon}_{\mathbf{X}}\\ \boldsymbol{\varepsilon}_{\mathbf{Y}} & -\boldsymbol{\varepsilon}_{\mathbf{X}} & \boldsymbol{\Delta}\mathbf{m}\end{bmatrix} . \begin{pmatrix}\overline{\mathbf{X}}\\ \overline{\mathbf{Y}}\\ \overline{\mathbf{Z}}\end{pmatrix}. \tag{5}$$

Now we symbolize $\boldsymbol{\omega} = (\mathbf{X_0},\ \mathbf{Y_0},\ \mathbf{Z_0}, \boldsymbol{\varepsilon}_{\mathbf{X}},\ \boldsymbol{\varepsilon}_{\mathbf{Y}},\ \boldsymbol{\varepsilon}_{\mathbf{Z}},\ \boldsymbol{\Delta}\mathbf{m})^{\mathbf{T}}$ as seven coordinate transformation parameters from the ITRF to the NSRS; $\tilde{\boldsymbol{\tau}}$ as vector of the adjusted spatial coordinates of the ground control points in the NSRS, which will be obtained after the combined adjustment of the terrestrial geodetic and passive GNSS networks and has the following form:

$$\tilde{\boldsymbol{\tau}} = \boldsymbol{\tau} + \boldsymbol{\delta\tau}. \tag{6}$$

where vector $\boldsymbol{\tau}$ is represented in form (3); $\boldsymbol{\delta\tau}$ - is vector of spatial coordinate corrections. $\widetilde{\mathbf{S}} = \mathbf{S} + \boldsymbol{\delta}\mathbf{S}$ as vector of the adjusted spatial coordinates of the GNSS points in the ITRF obtained after the combined adjustment of the terrestrial geodetic and passive GNSS networks. In addition, vector $\widetilde{\mathbf{S}}$ and vector of spatial coordinate corrections $\boldsymbol{\delta}\mathbf{S}$ have following forms with respect to vector $\mathbf{S}$ represented in form (4):

$$\widetilde{\mathbf{S}} = \begin{pmatrix} \widetilde{\mathbf{S}}_1 \\ \widetilde{\mathbf{S}}_2 \end{pmatrix}, \tag{7}$$

$$\delta\mathbf{S} = \begin{pmatrix} \boldsymbol{\delta}\mathbf{S}_1 \\ \boldsymbol{\delta}\mathbf{S}_2 \end{pmatrix}. \tag{8}$$

With above presented notations, for the np ground control points from formula (5) yields

$$\widetilde{\boldsymbol{\tau}} = \widetilde{\mathbf{S}}_1 + \mathbf{G}.\boldsymbol{\omega}. \tag{9}$$

where block matrix $\mathbf{G}$ with dimension NP × 7 has form:

$$\mathbf{G}_{\mathbf{NPx7}} = \begin{pmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \\ . \\ . \\ \mathbf{G}_{\mathbf{NP}} \end{pmatrix},$$

additionally sub-block matrix $\mathbf{G_i}$ with order 3 × 7 (i = 1,2,…,NP) is represented in following form:

$$\mathbf{G_i} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\overline{\mathbf{Z}}_\mathbf{i} & \overline{\mathbf{Y}}_\mathbf{i} & \overline{\mathbf{X}}_\mathbf{i} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \overline{\mathbf{Z}}_\mathbf{i} & \mathbf{0} & -\overline{\mathbf{X}}_\mathbf{i} & \overline{\mathbf{Y}}_\mathbf{i} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & -\overline{\mathbf{Y}}_\mathbf{i} & \overline{\mathbf{X}}_\mathbf{i} & \mathbf{0} & \overline{\mathbf{Z}}_\mathbf{i} \end{pmatrix}.$$

When the seven coordinate transformation parameters of Bursa-Wolf formula are unknown, the mathematical model of the combined adjustment of the terrestrial geodetic and GNSS network had been proposed in Ref. [31] in the following form:

$$\mathbf{V}_{\tau}^{kx1} = \boldsymbol{\delta\tau}^{kx1}, \quad \mathbf{P}_{\tau} = \boldsymbol{\mu}_{S}^{2}.\mathbf{K}_{\tau}^{-1},$$

$$\mathbf{V}_{S} = \boldsymbol{\delta}\mathbf{S}^{Kx1}, \quad \mathbf{P}_{S} = \mathbf{R}_{S}, \tag{10}$$

$$\boldsymbol{\delta}\mathbf{S}_{1}^{kx1} - \boldsymbol{\delta\tau}^{kx1} + \mathbf{G}.\boldsymbol{\omega} + \mathbf{L} = \mathbf{0},$$

where the third condition equation in the abovementioned model is inferred from the relation (9) accounting for formulas (6), (7), (8); vector of misclosures $\mathbf{L} = \mathbf{S_1} - \boldsymbol{\tau}$.

System of observation equations (10) has K + k + 7 unknown parameters, in which there are K + k spatial coordinate corrections.

In case the approximate values of the seven coordinate transformation parameters of Bursa-Wolf formula $\boldsymbol{\omega}^{(0)} = (\mathbf{X}_0^{(0)},\ \mathbf{Y}_0^{(0)},\ \mathbf{Z}_0^{(0)},\ \boldsymbol{\varepsilon}_X^{(0)},\ \boldsymbol{\varepsilon}_Y^{(0)},\ \boldsymbol{\varepsilon}_Z^{(0)},\ \boldsymbol{\Delta}\mathbf{m}^{(0)})^T$ had been determined, we fully can convert vector $\mathbf{S}$ (4) from the ITRF to the NSRS and get a vector of the transformed spatial coordinates $\boldsymbol{\theta}$ of the all GNSS points in the NSRS in the following form:

$$\boldsymbol{\theta} = \begin{pmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \end{pmatrix}, \tag{11}$$

where the subvector $\boldsymbol{\theta}_1$ corresponds to the np ground control points; subvector $\boldsymbol{\theta}_2$ refers to the remaining (NP – np) GNSS points.

In this case, a difference between the vector $\boldsymbol{\tau}$ (3) and the subvector $\boldsymbol{\theta}_1$ in (11) mainly was caused by the existence of errors in the vector $\boldsymbol{\tau}$ (3) and the vector of approximate seven coordinate transformation parameters $\boldsymbol{\omega}^{(0)}$. For the task of the combined adjustment of the terrestrial geodetic and passive GNSS networks in the NSRS, when we use the vector of the spatial coordinates $\boldsymbol{\theta}$ (11) as the vector of pseudo-measurements, an improvement in the accuracy of the national spatial coordinate vector $\boldsymbol{\tau}$ (3) will be obtained due to the high accuracy of the vector $\boldsymbol{\theta}$, large number of redundant pseudo-measurements and taking account of variance–covariance matrix $\mathbf{K}_S = \boldsymbol{\mu}_S^2.\mathbf{R}_S^{-1}$ of the vector $\boldsymbol{\theta}$.

We will carry out a research on the method of the combined adjustment of the terrestrial geodetic and passive GNSS networks in the NSRS proposed in [16]. In this method, the subvector $\boldsymbol{\theta}_2$ in the form of (11) will be used for the subvector of approximate spatial coordinate of the remaining (NP – np) GNSS points in the NSRS. Then taking into account vector $\boldsymbol{\tau}$ (3), the vector of the approximate spatial coordinate $\hat{\tau}$ of the all GNSS points in the NSRS has the following form:

$$\hat{\boldsymbol{\tau}} = \begin{pmatrix} \boldsymbol{\tau} \\ \boldsymbol{\theta}_2 \end{pmatrix}. \tag{12}$$

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Vector of spatial coordinate correction $\boldsymbol{\delta\hat{\tau}}$ and vector of last spatial coordinate $\tilde{\hat{\boldsymbol{\tau}}} = \hat{\boldsymbol{\tau}} + \boldsymbol{\delta\hat{\tau}}$ are represented in the following forms:

$$\boldsymbol{\delta\hat{\tau}} = \begin{pmatrix} \boldsymbol{\delta\tau} \\ \boldsymbol{\delta\tau}_{\mathbf{NP-np}} \end{pmatrix}, \tag{13}$$

$$\tilde{\hat{\boldsymbol{\tau}}} = \begin{pmatrix} \tilde{\boldsymbol{\tau}} \\ \tilde{\boldsymbol{\theta}}_{\mathbf{2}} \end{pmatrix}. \tag{14}$$

With the purpose of decrease in influence of the errors in the vector of approximate seven coordinate transformation parameters $\boldsymbol{\omega}^{(0)}$ on the results of the combined adjustment of the terrestrial geodetic and passive GNSS networks in the NSRS, we will use vector of corrections $\boldsymbol{\delta\omega} = \left(\boldsymbol{\delta X_0}, \boldsymbol{\delta Y_0}, \boldsymbol{\delta Z_0}\right)^{\mathbf{T}}$ applied to transformed coordinates by formula (5). For the vector of transformed spatial coordinates $\boldsymbol{\theta}$ (11), its last value $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} + \mathbf{V_S}$ is represented in the form:

$$\tilde{\boldsymbol{\theta}} = \begin{pmatrix} \tilde{\boldsymbol{\theta}}_{\mathbf{1}} \\ \tilde{\boldsymbol{\theta}}_{\mathbf{2}} \end{pmatrix}, \tag{15}$$

where $\mathbf{V_S}$ is the vector of corresponding spatial coordinate corrections.

From the relation

$$\tilde{\boldsymbol{\theta}} + \boldsymbol{\Omega}.\boldsymbol{\delta\omega} = \tilde{\hat{\boldsymbol{\tau}}},$$

where the block matrix $\boldsymbol{\Omega}$ with dimension NP x 3 has the form:

$$\boldsymbol{\Omega}_{\mathbf{NPx3}} = \begin{pmatrix} \mathbf{E_1} \\ \mathbf{E_2} \\ \cdot \\ \cdot \\ \mathbf{E_{NP}} \end{pmatrix}, \tag{16}$$

additionally sub-block matrix $\mathbf{E_i}$ is an unit matrix of the order of 3x3 (i = 1,2,..,NP), taking into account the formulas (11), (12), (13), (14), (15). We obtain the system of observation equations in the following form:

$$\mathbf{V_S^{Kx1}} = \boldsymbol{\delta\hat{\tau}}_{\mathbf{S}}^{\mathbf{Kx1}} - \boldsymbol{\Omega}.\boldsymbol{\delta\omega}^{\mathbf{3x1}} + \mathbf{L_S^{Kx1}},$$

where the vector of free components has the form:

$$\mathbf{L}_S^{Kx1} = \hat{\boldsymbol{\tau}} - \boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\tau} - \boldsymbol{\theta}_1 \\ \text{..........} \; \text{.......} \\ \mathbf{O} \end{bmatrix},$$

O is the subvector corresponding to the subvector $\boldsymbol{\theta}_2$ and containing (K – k) zeros.

Finally, we obtain the mathematical model of the combined adjustment of the terrestrial geodetic and passive GNSS networks in the NSRS in the following form [16, 20]:

$$\begin{aligned} &\mathbf{V}_\tau^{kx1} = \boldsymbol{\delta\tau}^{kx1}, \quad \mathbf{P}_\tau = \boldsymbol{\mu}_S^2.\mathbf{K}_\tau^{-1}, \\ &\mathbf{V}_S^{Kx1} = \boldsymbol{\delta}\hat{\boldsymbol{\tau}}_S^{Kx1} - \boldsymbol{\Omega}.\boldsymbol{\delta\omega}^{3x1} + \mathbf{L}_S^{Kx1}, \quad \mathbf{P}_S = \mathbf{R}_S, \end{aligned} \tag{17}$$

where the vector of spatial coordinate corrections $\boldsymbol{\delta}\hat{\boldsymbol{\tau}}$ has the form (13).

It should be underlined that at present we can determine the seven coordinate transformation parameters of Bursa-Wolf formula $\boldsymbol{\omega}^{(0)} = (\mathbf{X}_0^{(0)}, \; \mathbf{Y}_0^{(0)}, \; \mathbf{Z}_0^{(0)}, \; \boldsymbol{\varepsilon}_X^{(0)}, \; \boldsymbol{\varepsilon}_Y^{(0)}, \; \boldsymbol{\varepsilon}_Z^{(0)}, \boldsymbol{\Delta}\mathbf{m}^{(0)})^T$ with very high accuracy. In this case, the variance-covariance matrix $\mathbf{K}_S = \boldsymbol{\mu}_S^2.\mathbf{R}_S^{-1}$, obtained after the separate adjustment of the passive GNSS network in the ITRF is considered to be unchanged in the process of the transformation of spatial coordinates of GNSS points from the ITRF into the NSRS. Therefore, the weight matrix $\mathbf{P}_S = \mathbf{R}_S$ is assigned to the second subsystem of observation equations in (17).

System of observation equations (17) has all K + 3 unknown parameters. A study of the method of Givens rotation for solving this system of observation equations is performed in Subsection 2.4.

2.2. Brief description of the method of recurrent adjustment of geodetic network with Givens rotation

To obtain the best linear unbiased estimate of unknown parameters by the least squares method, we must adopt an outlier detection method for geodetic observations in geodetic networks. In [29], a method of recurrent adjustment of geodetic networks had been developed, which allows for the detection of outliers in the calculation process and is realized by the following procedure: A recurrent adjustment process is performed sequentially for every measured value in combination with outlier detection method for redundant measurements. Because the method of recurrent adjustment is working with an inverse matrix **Q** related to a normal matrix **R** by the formula $\mathbf{Q} = \mathbf{R}^{-1}$, this method is called as "Q – recurrent algorithm."

First, we will investigate the method of recurrent adjustment of geodetic networks containing n independent measurements and k unknown parameters. For the *i*th measured value $\mathbf{y}_i$ (i = 1,2,…,n), its adjusted value $\tilde{\mathbf{y}}_i = \mathbf{y}_i + \mathbf{v}_i$ is related to the adjusted vector of unknown

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parameters $\tilde{\mathbf{X}} = \mathbf{X}^{(0)} + \boldsymbol{\delta}\mathbf{X}$ by a function $\tilde{\mathbf{y}}_\mathbf{i} = \varphi_\mathbf{i}(\tilde{\mathbf{X}})$, where $\mathbf{v}_\mathbf{i}$ - is correction (residual) to the *i*th measured value $\mathbf{y}_\mathbf{i}$; $\mathbf{X}^{(0)}$ - is vector of approximate values of the unknown parameters with dimension k × 1; $\boldsymbol{\delta}\mathbf{X}$ - is vector of corrections to the vector $\mathbf{X}^{(0)}$ with dimension k × 1; k – number of unknown parameters.

After performing the Taylor linear expansion, we obtain the observation equation of the *i*th measurement $\mathbf{y}_\mathbf{i}$ in the following form:

$$\mathbf{v}_\mathbf{i} = \mathbf{a}_\mathbf{i}.\delta\mathbf{X}_\mathbf{i} + \mathbf{l}_\mathbf{i}^{(0)}, \tag{18}$$

according to weight $\mathbf{p}_\mathbf{i}$, where $\mathbf{a}_\mathbf{i}$ - row vector of coefficients with dimension 1 × k; $\mathbf{l}_\mathbf{i}^{(0)} = \varphi_\mathbf{i}(\mathbf{X}^{(0)}) - \mathbf{y}_\mathbf{i}$ - free component.

For the every *i*th measured value $\mathbf{y}_\mathbf{i}$, inserted in recurrent adjustment process, we will calculate an inverse matrix $\mathbf{Q}_\mathbf{i}$ of the order of k x k, vector of corrections $\boldsymbol{\delta}\mathbf{X}_\mathbf{i}$ and value $\boldsymbol{\Phi}_\mathbf{i} = [\mathbf{V}^\mathbf{T}\mathbf{PV}]_\mathbf{i}$.

To start the recurrent adjustment process, we obtain:

the initial inverse matrix $\mathbf{Q}_\mathbf{0} = \mathbf{10}^\mathbf{m}.\mathbf{E}_\mathbf{kxk}$,

initial vector of corrections $\boldsymbol{\delta}\mathbf{X}_\mathbf{0} = \mathbf{0}$,

initial value $\boldsymbol{\Phi}_\mathbf{0} = [\mathbf{V}^\mathbf{T}\mathbf{PV}]_\mathbf{0} = \mathbf{0}$,

where the number m is equal to 6 and $\mathbf{E}_\mathbf{kxk}$ is the identity matrix of the order of k × k.

It is assumed that after performing the recurrent adjustment process for (i − 1) first measured values, we have obtained the inverse matrix $\mathbf{Q}_\mathbf{i-1}$, vector of corrections $\boldsymbol{\delta}\mathbf{X}_\mathbf{i-1}$ and value $\boldsymbol{\Phi}_\mathbf{i-1} = [\mathbf{V}^\mathbf{T}\mathbf{PV}]_\mathbf{i-1}$. The recurrent adjustment process for the *i*th measurement $\mathbf{y}_\mathbf{i}$ with the observation equation (18) will be performed by the following way:

$$\mathbf{Q}_\mathbf{i} = \mathbf{Q}_\mathbf{i-1} - \frac{\mathbf{Z}_\mathbf{i}^\mathbf{T}\mathbf{Z}_\mathbf{i}}{\mathbf{g}_\mathbf{i}},$$

$$\boldsymbol{\delta}\mathbf{X}_\mathbf{i} = \boldsymbol{\delta}\mathbf{X}_\mathbf{i-1} - \frac{\mathbf{Z}_\mathbf{i}^\mathbf{T}}{\mathbf{g}_\mathbf{i}}\mathbf{l}_\mathbf{i},$$

$$\boldsymbol{\Phi}_\mathbf{i} = \boldsymbol{\Phi}_\mathbf{i-1} + \frac{\mathbf{l}_\mathbf{i}^\mathbf{2}}{\mathbf{g}_\mathbf{i}},$$

where the vector

$$\mathbf{Z_i}^{\mathbf{T}} = \mathbf{Q_{i-1}}.\mathbf{a_i}^{\mathbf{T}},$$

the free component

$$\mathbf{l_i} = \mathbf{a_i}.\delta\mathbf{X}_{\mathbf{i-1}} + \mathbf{l_i}^{(0)}. \tag{19}$$

The number $\mathbf{g_i}$ is an inverse weight of the free component $\mathbf{l_i}$ and is calculated by the formula:

$$\mathbf{g_i} = \mathbf{p_i}^{-1} + \mathbf{a_i}.\mathbf{Z_i}^{\mathbf{T}}. \tag{20}$$

The *i*th measurement $\mathbf{y_i}$ will be recognized as the redundant measurement, if number $\mathbf{g_i}$ satisfies the condition $\mathbf{g_i} \le \frac{\mathbf{100}}{\mathbf{p_i}}$ [30]. When $\mathbf{y_i}$ is the redundant measurement, the outlier detection will be performed based on the comparison of the free component $\mathbf{l_i}$ with its limitation $(\mathbf{l_i})_{\mathbf{lim}} = \mathbf{3}.\mu_{\mathbf{0}}.\sqrt{\mathbf{g_i}}$, where $\mu_{\mathbf{0}}$ - is the RMS error of measurements determined apriori. If $(\mathbf{l_i})_{\mathbf{lim}} > \mathbf{l_i}$, then we have base to accept an assumption that in the first i measured values outliers exist.

In the case of the absence of any outliers in the geodetic network, after accomplishment of the recurrent adjustment process for n measurements, the vector of adjusted parameters $\tilde{\mathbf{X}}$ and the RMS error of weight unit μ after adjustment of the geodetic network have been calculated by the following formulas:

$$\tilde{\mathbf{X}} = \mathbf{X}^{(0)} + \delta\mathbf{X_n}, \tag{21}$$

$$\mu = \pm\sqrt{\frac{\Phi_{\mathbf{n}}}{\mathbf{n}-\mathbf{k}}}. \tag{22}$$

Although the recurrent algorithm Q has the ability to detect outliers in recurrent adjustment process, the inverse matrix **Q** is a full matrix that leads to a decrease in the efficiency of the adjustment of a large geodetic network. The method of Givens rotation becomes efficient in case of using a sparse matrix technique [12]. In [13], the usage of Givens rotation method had been proposed for the adjustment of large geodetic networks. The method of Givens rotation allows the transformation of the elements of the coefficients matrix $\mathbf{A_{nxk}}$ in the system of observation equations to the elements of an upper triangular matrix $\mathbf{T_{kxk}}$ related to the normal matrix **R** by the formula $\mathbf{R} = \mathbf{T}^{\mathbf{T}}\mathbf{T}$.

On an account of abilities of the method of recurrent adjustment for outlier detection in recurrent adjustment process and the method of Givens rotation for using the technique of a sparse matrix, in [15], a method of recurrent adjustment with rotation that had been constructed based on the method of Givens rotation had been proposed by using the technique of sparse matrix and has been performed in the procedure of recurrent adjustment process with outlier detection. This method is called as "T – recurrent algorithm" with an initial matrix of $\mathbf{T_0}$ of the recurrent adjustment process represented in the following form:

$$\mathbf{T_0} = \mathbf{10^{-m}.E_{kxk}}, \tag{23}$$

where the number m is equal to 6; $\mathbf{E_{kxk}}$- is identity matrix of order k; k is number of unknown parameters.

It is necessary to underline that for the method of Givens rotation, a transformation of every element of the row vector of coefficients $\mathbf{a_i}$ in the observation equation (18) requires four multiplications. For a method of fast rotation proposed in [10], the transformation of every element of the row vector of coefficients $\mathbf{a_i}$ in the observation equation (18) requires two multiplications. However using the initial matrix $\mathbf{T_0}$ (23) for starting the recurrent adjustment process, the method of fast rotation leads to an increase of the transformed elements of the upper triangular matrix $\mathbf{T_{kxk}}$. That is why in [17] it had been proposed that the method of mean rotation for that in the recurrent adjustment process the transformation of every element of the row vector of coefficients $\mathbf{a_i}$ in the observation equation (18) requires three multiplications. For the method of mean rotation, the upper triangular matrix $\mathbf{T_{kxk}}$ is represented in the form $\mathbf{T_{kxk}} = \mathbf{D}.\hat{\mathbf{T}}$, where $\mathbf{D}$ is a diagonal matrix containing diagonal elements of the upper triangular matrix $\mathbf{T_{kxk}}$, $\hat{\mathbf{T}}$ is an upper triangular matrix with unit diagonal elements.

In this chapter, we carry out a research on the usage of T - recurrent algorithm for the recurrent adjustment of geodetic networks containing n independent values of measurements. We symbolize $\mathbf{Y}$ as the vector of transformed free components related to the vector of corrections $\boldsymbol{\delta}\mathbf{X}$ by the system of equations.

$$\mathbf{T}.\delta\mathbf{X} = \mathbf{Y}. \tag{24}$$

For starting the recurrent adjustment process, we get the initial matrix $\mathbf{T_0}$ form (23), initial vector of transformed free components $\mathbf{Y_0} = \mathbf{0}$ and initial value $\mathbf{\Phi_0} = [\mathbf{V^T PV}]_\mathbf{0} = \mathbf{0}$. It is assumed that after performing the recurrent adjustment process for the first (i − 1) values of measurements, we have obtained an upper triangular matrix $\mathbf{T_{i-1}}$, a vector of transformed free components $\mathbf{Y_{i-1}}$ and a value $\mathbf{\Phi_{i-1}} = [\mathbf{V^T PV}]_\mathbf{i-1}$.

For sequential insertion of the *i*th measured value $\mathbf{y_i}$ with the observation equation (18) in the recurrent adjustment process, we will create auxiliary matrix $\mathbf{B^{(0)}}$ with dimensions (k + 1) × (k + 1) in the following form ([15], [18]):

$$\mathbf{B}^{(0)} = \left[\begin{array}{c:c} \mathbf{T}_{i-1} & \mathbf{Y}_{i-1} \\ \hdashline \sqrt{\mathbf{p}_i}\cdot \mathbf{a}_i & \sqrt{\mathbf{p}_i}\cdot \mathbf{l}_i^{(0)} \end{array}\right] \tag{25}$$

We symbolize $\mathbf{b}_j$ $(j = 1,2,.., k)$ as jth row of matrix $\mathbf{B}^{(0)}$ (25), $\xi^{(0)}$ as $(k+1)$th row of matrix $\mathbf{B}^{(0)}$ (25). A rotation transformation will be sequentially performed from row $\mathbf{b}_1$ to row $\mathbf{b}_k$ of the matrix $\mathbf{B}^{(0)}$. It is assumed that after performing rotation transformation on first (j – 1) rows, we have got the matrix $\mathbf{B}^{(j-1)}$ with (j – 1) transformed rows and transformed $(k+1)$th row $\xi^{(j-1)}$. For the rotation transformation of jth row $\mathbf{b}_j$ of matrix $\mathbf{B}^{(j-1)}$, we build a rotation matrix $\mathbf{H}_j$ in underrepresented form (26). The elements $\mathbf{C}_j$ and $\mathbf{S}_j$ of the rotation matrix $\mathbf{H}_j$ are calculated by the following formulas:

$$\mathbf{C}_j = \frac{(\mathbf{b}_j)_j}{\mathbf{f}}, \quad \mathbf{S}_j = -\frac{\xi_j^{(j-1)}}{\mathbf{f}},$$

where $(\mathbf{b}_j)_j$ is the jth element of row $\mathbf{b}_j$; $\xi_j^{(j-1)}$ is the jth element of the $(k+1)$th row $\xi^{(j-1)}$;

$$\mathbf{f} = \sqrt{\left\{(\mathbf{b}_j)_j\right\}^2 + \left\{\xi_j^{(j-1)}\right\}^2}.$$

The elements $\mathbf{C}_j$ and $\mathbf{S}_j$ of the rotation matrix $\mathbf{H}_j$ are located on the jth and $(k+1)$th rows as well as on the jth and $(k+1)$th columns as represented in form (26).

Multiplying matrix $\mathbf{B}^{(j-1)}$ on the left by the rotation matrix $\mathbf{H}_j$, we will obtain the transformed matrix $\mathbf{B}^{(j)}$ that is.

$$\mathbf{B}^{(j)} = \mathbf{H}_j.\mathbf{B}^{(j-1)} = \mathbf{H}_j.\mathbf{H}_{j-1}...\mathbf{H}_1.\mathbf{B}^{(0)}.$$

$$\mathbf{H}_j = \begin{bmatrix} 1 & 0 & .. & 0 & .. & 0 & .. & 0 \\ 0 & 1 & .. & 0 & .. & 0 & .. & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & .. & \mathbf{C}_j & .. & 0 & .. & -\mathbf{S}_j \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & ... & 0 & .. & 1 & .. & 0 \\ 0 & 0 & ... & \mathbf{S}_j & .. & 0 & .. & \mathbf{C}_j \end{bmatrix}_{(k+1)\times(k+1)} \tag{26}$$

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By such a way after the accomplishment of rotation transformation of all k rows of the matrix $\mathbf{B}^{(0)}$(25), we obtain the transformed matrix

$$\widetilde{\mathbf{B}} = \mathbf{B}^{(k)} = \mathbf{H_k} \cdot \mathbf{H_{k-1}} \cdots \mathbf{H_1} \cdot \mathbf{B}^{(0)}, \tag{27}$$

which has the following form:

$$\widetilde{\mathbf{B}} = \begin{bmatrix} \mathbf{T_i} & \mathbf{Y_i} \\ \mathbf{0} & \sqrt{\Delta\mathbf{\Phi}_i} \end{bmatrix}. \tag{28}$$

When $\mathbf{\Phi}_i = [\mathbf{V^T PV}]_i = \mathbf{\Phi}_{i-1} + \Delta\mathbf{\Phi}_i$. With the purpose of the outlier detection for the *i*th measurement $\mathbf{y_i}$, which is a redundant measured value, we will calculate a vector $\mathbf{t_i}$ from the system $\mathbf{T}_{i-1}^{\mathbf{T}} \cdot \mathbf{t_i} = \mathbf{a}_i^{\mathbf{T}}$. The free component $\mathbf{l_i}$ (19) and its inverse weight $\mathbf{g_i}$ (20) are calculated by the following formulas [15, 18]:

$$\mathbf{l_i} = \mathbf{t}_i^{\mathbf{T}} \cdot \mathbf{Y}_{i-1} + \mathbf{l}_i^{(0)},$$
$$\mathbf{g_i} = \mathbf{p}_i^{-1} + \mathbf{t}_i^{\mathbf{T}} \cdot \mathbf{t_i}.$$

The outlier detection will then be performed by a way, analogous to the Q – recurrent algorithm. After the accomplishment of the recurrent adjustment process for the n measured values, the vector of corrections $\boldsymbol{\delta}\mathbf{X}$ is calculated from the system of equations (24). The vector of adjusted parameters $\widetilde{\mathbf{X}}$ and the RMS error of weight unit $\boldsymbol{\mu}$ after the adjustment of the geodetic network are then calculated by formulas (21), (22).

The correctness of the form (28), obtained from Givens rotation, can be checked by the following way [15, 18]. It is assumed that for the first (i-1) measured values in a geodetic network, we have a system of observation equations in the following form:

$$\mathbf{V}_{i-1} = \mathbf{A}_{i-1}.\delta\mathbf{X}_{i-1} + \mathbf{L}_{i-1}^{(0)} \tag{29}$$

with a weight matrix $\mathbf{P}_{i-1}$.

Solving the system of observation equations (29) by the least squares method, we obtain the system of normal equations:

$$\mathbf{R}_{i-1}.\delta\mathbf{X}_{i-1} + \mathbf{b}_{i-1} = \mathbf{0}, \tag{30}$$

where $\mathbf{R}_{i-1} = \mathbf{A}_{i-1}^{\mathbf{T}}.\mathbf{P}_{i-1}.\mathbf{A}_{i-1}$ is the normal matrix and $\mathbf{b}_{i-1} = \mathbf{A}_{i-1}^{\mathbf{T}}.\mathbf{P}_{i-1}.\mathbf{L}_{i-1}^{(0)}$ is the vector of free components of the system of normal equations.

After the Cholesky decomposition, the system of normal equations (30) has been transformed into a system of equivalent equations:

$$\mathbf{T}_{i-1}^{T}.\mathbf{Y}_{i-1} = -\mathbf{b}_{i-1}, \\ \mathbf{T}_{i-1}.\delta\mathbf{X}_{i-1} = \mathbf{Y}_{i-1}, \tag{31}$$

where $\mathbf{Y}_{i-1}$ is the vector of transformed free components, $\mathbf{T}_{i-1}$ is the upper triangular matrix obtained from the Cholesky decomposition $\mathbf{R}_{i-1} = \mathbf{T}_{i-1}^{T}.\mathbf{T}_{i-1}$.

From formula (31), we can obtain the following value:

$$(\delta\mathbf{X}_{i-1}^{T}.\mathbf{b}_{i-1})^{T} = \delta\mathbf{X}_{i-1}^{T}.\mathbf{b}_{i-1} = -\mathbf{Y}_{i-1}^{T}.\mathbf{Y}_{i-1}. \tag{32}$$

On an account of the formulas (30) and (32) from (29), we will obtain a following value:

$$\mathbf{\Phi}_{i-1} = \mathbf{V}_{i-1}^{T}.\mathbf{P}_{i-1}.\mathbf{V}_{i-1} = \left(\mathbf{L}_{i-1}^{(0)}\right)^{T}.\mathbf{P}_{i-1}.\mathbf{L}_{i-1}^{(0)} - \mathbf{Y}_{i-1}^{T}.\mathbf{Y}_{i-1}. \tag{33}$$

Now after insertion of the *i*th measured value $\mathbf{y}_i$ with the observation equation (18) in the adjustment process, we will obtain some known relations:

$$\mathbf{R}_i = \mathbf{T}_i^{T}.\mathbf{T}_i = \mathbf{R}_{i-1} + \mathbf{p}_i.\mathbf{a}_i^{T}.\mathbf{a}_i = \mathbf{T}_{i-1}^{T}.\mathbf{T}_{i-1} + \mathbf{p}_i.\mathbf{a}_i^{T}.\mathbf{a}_i, \\ \mathbf{T}_i^{T}.\mathbf{Y}_i = -\mathbf{b}_i = -\mathbf{b}_{i-1} - \mathbf{p}_i.\mathbf{a}_i^{T}.\mathbf{l}_i^{(0)} = \mathbf{T}_{i-1}^{T}.\mathbf{Y}_{i-1} - \mathbf{p}_i.\mathbf{a}_i^{T}.\mathbf{l}_i^{(0)}. \tag{34}$$

By an analogous way to formula (33), we have

$$\mathbf{\Phi}_i = \mathbf{V}_i^{T}.\mathbf{P}_i.\mathbf{V}_i = \left(\mathbf{L}_i^{(0)}\right)^{T}.\mathbf{P}_i.\mathbf{L}_i^{(0)} - \mathbf{Y}_i^{T}.\mathbf{Y}_i. \tag{35}$$

On an account of the relation $\left(\mathbf{L}_i^{(0)}\right)^{T}.\mathbf{P}_i.\mathbf{L}_i^{(0)} = \left(\mathbf{L}_{i-1}^{(0)}\right)^{T}.\mathbf{P}_{i-1}.\mathbf{L}_{i-1}^{(0)} + \mathbf{p}_i.(\mathbf{l}_i^{(0)})^2$, from formulas (33) and (35) will be inferred the following value:

$$\Delta\mathbf{\Phi}_i = \mathbf{\Phi}_i - \mathbf{\Phi}_{i-1} = \mathbf{p}_i.(\mathbf{l}_i^{(0)})^2 - \mathbf{Y}_i^{T}.\mathbf{Y}_i + \mathbf{Y}_{i-1}^{T}.\mathbf{Y}_{i-1}. \tag{36}$$

Because the rotation matrix $\mathbf{H}_j$(26) is the orthogonal matrix that satisfies the condition $\mathbf{H}_j^{T}.\mathbf{H}_j = \mathbf{E}_{(k+1)x(k+1)}$, where $\mathbf{E}_{(k+1)x(k+1)}$ is the unit matrix of the order of (k + 1) × (k + 1), from formula (27), we obtain the following relationship:

$$\tilde{\mathbf{B}}^{T}.\tilde{\mathbf{B}} = (\mathbf{B}^{(0)})^{T}.\mathbf{B}^{(0)}. \tag{37}$$

Substituting $\mathbf{B}^{(0)}$ (25) and $\tilde{\mathbf{B}}$ (28) into (37), we obtain the known formulas (34) and (36). That proved the correctness of the form (28), obtained from Givens rotation after the insertion of the *i*th measured value $\mathbf{y}_i$ with the observation equation (18) in the recurrent adjustment process.

In the case outliers exist in the geodetic network, we will determine the corrections vector $\mathbf{V}^{(0)}$ for n measurements that will be used for finding outliers. A method for finding outliers is investigated in Subsection 2.3.

2.3. Method for finding outliers in the geodetic network

In case the dispersion σ of measurements has not been derived confidently and has been changed in whole measurement process, i. e. $0 \le \sigma < \infty$, errors of measurements obey a Laplace distribution [32]. In this case apart from random errors, errors of measurements still consist of gross errors and as the maximum likelihood estimate, the least absolute residuals (LAR) estimate will be established under the following L_1 - norm condition:

$$\sum_{i=1}^{n} \left| \bar{\mathbf{v}}_{\mathbf{i}} \right| = \mathbf{min}, \tag{38}$$

where $\bar{\mathbf{v}}_{\mathbf{i}} = \sqrt{\mathbf{p}_{\mathbf{i}}}.\mathbf{v}_{\mathbf{i}}$; $\mathbf{p}_{\mathbf{i}}$ is the weight of *i*th measurement $\mathbf{y}_{\mathbf{i}}$;$\mathbf{v}_{\mathbf{i}}$ is the correction (residual) to this *i*th measurement and i = 1,2,…,n.

The LAR method is more efficient in estimating the parameters of the regression model; in the case, the data are contaminated with gross errors. The LAR method has the ability of resisting against blunders (outliers) [39]. Accounting for the popularity of the calculation schema by the least squares method, in [11] had been proposed an iteratively reweighted least squares (IRLS) method, through which condition (38) is represented in the form:

$$\sum_{i=1}^{n} \left| \bar{\mathbf{v}}_{\mathbf{i}} \right| = \sum_{i=1}^{n} \bar{\mathbf{p}}_{\mathbf{i}} \cdot \bar{\mathbf{v}}_{\mathbf{i}}^{2} = \mathbf{min}, \tag{39}$$

where weight $\bar{\mathbf{p}}_{\mathbf{i}} = \dfrac{\mathbf{1}}{\left|\bar{\mathbf{v}}_{\mathbf{i}}\right|}$.

In [5], a convergence of the iterative calculation process by the IRLS method and a diminution of amplitude of absolute residuals after every iteration under the condition had been proven (39). The experiments show that the IRLS method allows outliers to be found reliably only for such dense geodetic networks with large number of redundant measurements such as traditional triangulation, the GNSS network and the vertical network created by leveling lines between nodal benchmarks [18].

First, we symbolize $\overline{m}$ as the number of iterations ($\overline{m}$ = 0,1,2,...). As presented in Subsection 2.2, after adjusting the geodetic network by the T- recurrent algorithm with the discovery of existence of outliers in the geodetic network, we have calculated the vector $\mathbf{V}^{(0)}$ of corrections to n measurements that will be used for the iterative adjustment of the geodetic network by the IRLS method in order to find outliers. In the $\overline{m}$th iteration, based on the condition (39) for the *i*th measurement $\mathbf{y}_{\mathbf{i}}$ the observation equation (18) will be expressed in the following form:

$$\bar{\mathbf{v}}_{\mathbf{i}}^{(\bar{\mathbf{m}})} = \bar{\mathbf{a}}_{\mathbf{i}}.\delta\mathbf{X}_{\mathbf{i}}^{(\bar{\mathbf{m}})} + \bar{\mathbf{l}}_{\mathbf{i}}^{(0)} \tag{40}$$

with weight $\bar{\mathbf{p}}_{\mathbf{i}}^{(\bar{\mathbf{m}})} = \dfrac{\mathbf{1}}{\left|\bar{\mathbf{v}}_{\mathbf{i}}^{(\bar{\mathbf{m}}-1)}\right|}$, where $\bar{\mathbf{a}}_{\mathbf{i}} = \sqrt{\mathbf{p}_{\mathbf{i}}}.\mathbf{a}_i$ is the row vector of coefficients; $\bar{\mathbf{l}}_{\mathbf{i}}^{(0)} = \sqrt{\mathbf{p}_{\mathbf{i}}}.\mathbf{l}_{\mathbf{i}}^{(0)}$ is the free component; $\bar{\mathbf{v}}_{\mathbf{i}}^{(\bar{\mathbf{m}}-1)} = \sqrt{\mathbf{p}_{\mathbf{i}}}.\mathbf{v}_{\mathbf{i}}^{(\bar{\mathbf{m}}-1)}$, and $\mathbf{v}_{\mathbf{i}}^{(\bar{\mathbf{m}}-1)}$ are the correction for the ith measurement $\mathbf{y}_{\mathbf{i}}$ which is obtained in previous ($\bar{m}-1$) th iteration; $\mathbf{p}_{\mathbf{i}}$ is the weight of the ith measurement $\mathbf{y}_{\mathbf{i}}$.

The observation equation (40) will be sequentially inserted in the recurrent adjustment process by the T- recurrent algorithm. After the accomplishment of $\bar{m}$th iterative recurrent adjustment of the geodetic network with n measured values, we will calculate the vector of the adjusted parameters $\tilde{\mathbf{X}}^{(\bar{\mathbf{m}})}$ in the $\bar{m}$th iteration by the formula $\tilde{\mathbf{X}}^{(\bar{\mathbf{m}})} = \mathbf{X}^{(0)} + \delta\mathbf{X}^{(\bar{\mathbf{m}})}$. The vector $\tilde{\mathbf{X}}^{(\bar{\mathbf{m}})}$ will be used for the determination of the vector $\mathbf{V}^{(\bar{\mathbf{m}})}$ of corrections to n measured values serving next ($\bar{m}$ +1) th iterative recurrent adjustment of the geodetic network.

A process of the iterative recurrent adjustment of the geodetic network will be ended, if in two ($\bar{m}$ -1)th and $\bar{m}$th adjacent iterations for all residuals satisfy the following condition:

$$\left\| \left|\mathbf{v}^{(\mathbf{m})}\right| - \left|\mathbf{v}^{(\mathbf{m}-1)}\right| \right\| \leq \boldsymbol{\varepsilon},$$

where $\boldsymbol{\varepsilon}$ is a small positive number. The outliers can be found from the measured values which have the largest residuals (corrections).

2.4. Application of the recurrent adjustment method with Givens rotation for separate adjustment of GNSS network in the ITRF and next its combination to the NSRS

For the GNSS network comprising NP GNSS points, the components $\Delta\mathbf{X}, \ \Delta\mathbf{Y}, \ \Delta\mathbf{Z}$ of baseline vectors are used as pseudo-observations for the adjustment of this network. It is assumed that the GNSS network contains N baseline vectors. We symbolize $\bar{\mathbf{Y}}_{\mathbf{i}}$ ($i = 1,2,..., N$) as the vector of pseudo-observations between two GNSS points $\mathbf{s},\mathbf{h}$. Additionally,

$$\bar{\mathbf{Y}}_{\mathbf{i}} = \begin{bmatrix} \Delta\mathbf{X}_{\mathbf{i}} \\ \Delta\mathbf{Y}_{\mathbf{i}} \\ \Delta\mathbf{Z}_{\mathbf{i}} \end{bmatrix} \tag{41}$$

with variance-covariance matrix $\mathbf{C}_{\mathbf{i}}$ of the order of 3. That means that $\Delta\mathbf{X}_{\mathbf{i}}, \ \Delta\mathbf{Y}_{\mathbf{i}}, \ \Delta\mathbf{Z}_{\mathbf{i}}$ are dependent observations to which the system of observation equations corresponds in the following form:

$$\mathbf{V_i} = \mathbf{A_i}.\delta\mathbf{H_i} + \mathbf{L_i}^{(0)}, \tag{42}$$

where $\mathbf{V_i}$ is a vector of corrections (residuals) to the measured values $\mathbf{\Delta X_i}$, $\mathbf{\Delta Y_i}$, $\mathbf{\Delta Z_i}$ in the vector of pseudo-observations $\overline{\mathbf{Y}}_\mathbf{i}$ (41). The matrix of coefficients with dimension 3 x K (K = 3. NP – total number of unknown parameters in the GNSS network) has the form:

$$\mathbf{A_i} = \begin{bmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \mathbf{a_3} \end{bmatrix},$$

additionally $\mathbf{a_1}$ = **(–1 0 0 ... 1 0 0),** $\mathbf{a_2}$ = **(0 –1 0 ... 0 1 0),** $\mathbf{a_3}$ = **(0 0 –1... 0 0 1)**; $\delta\mathbf{H_i}$ is a vector of unknown corrections to approximate spatial coordinates of GNSS points in the ITRF, obtained after the insertion of *i*th vector of pseudo-observations $\overline{\mathbf{Y}}_\mathbf{i}$ (41) in the recurrent adjustment process; $\mathbf{L_i}^{(0)}$ is a vector of free components which has form:

$$\mathbf{L_i}^{(0)} = \begin{bmatrix} \mathbf{X_h}^{(0)} - \mathbf{X_s}^{(0)} - \mathbf{\Delta X_i} \\ \mathbf{Y_h}^{(0)} - \mathbf{Y_s}^{(0)} - \mathbf{\Delta Y_i} \\ \mathbf{Z_h}^{(0)} - \mathbf{Z_s}^{(0)} - \mathbf{\Delta Z_i} \end{bmatrix},$$

where $\mathbf{X_s}^{(0)}, \mathbf{Y_s}^{(0)}, \mathbf{Z_s}^{(0)}$ and $\mathbf{X_h}^{(0)}, \mathbf{Y_h}^{(0)}, \mathbf{Z_h}^{(0)}$ are the approximate spatial coordinates of the GNSS **s** and **h**.

A weight matrix $\mathbf{P_i}$ of the order 3 is assigned to the vector of pseudo-observations $\overline{Y}_i$ (41) and represented in form:

$$\mathbf{P_i} = \boldsymbol{\mu}_\mathbf{0}^\mathbf{2} \cdot \mathbf{C_i}^{-1}, \tag{43}$$

where $\boldsymbol{\mu}_\mathbf{0}$ is the RMS of unit weight determined apriori.

As we had seen in Subsection 2.2, with the purpose of outlier detection, the recurrent adjustment method is effectively realized for independent observations. The components $\mathbf{\Delta X_i}$, $\mathbf{\Delta Y_i}$, $\mathbf{\Delta Z_i}$ are the dependent observations. Therefore, for the application of the recurrent adjustment method, we must transform the dependent observations $\mathbf{\Delta X_i}$, $\mathbf{\Delta Y_i}$, $\mathbf{\Delta Z_i}$ to the independent ones. For that, we represent the weight matrix $\mathbf{P_i}$ in the form $\mathbf{P_i} = \mathbf{U_i^T}.\mathbf{U_i}$, and the system of observation equations (42) will be expressed as [20]:

$$\overline{\mathbf{V}}_{\mathbf{i}} = \overline{\mathbf{A}}_{\mathbf{i}}.\delta\mathbf{H}_{\mathbf{i}} + \overline{\mathbf{L}}_{\mathbf{i}}^{(\mathbf{0})}, \tag{44}$$

where $\overline{\mathbf{V}}_{\mathbf{i}} = \mathbf{U}_{\mathbf{i}}.\mathbf{V}_{\mathbf{i}}$, $\overline{\mathbf{A}}_{\mathbf{i}} = \mathbf{U}_{\mathbf{i}}.\mathbf{A}_{\mathbf{i}}$, $\overline{\mathbf{L}}_{\mathbf{i}}^{(\mathbf{0})} = \mathbf{U}_{\mathbf{i}}.\mathbf{L}_{\mathbf{i}}^{(\mathbf{0})}$.

By such a way, the system of observation equations (44) has a unit weight matrix $\overline{\mathbf{P}}_{\mathbf{i}} = \mathbf{E}_{\mathbf{3x3}}$, where $\mathbf{E}_{\mathbf{3x3}}$ is the unit matrix of the order of 3 x 3. In [20], an algorithm for a transformation of a matrix $\frac{\mathbf{1}}{\mu_{\mathbf{0}}^{\mathbf{2}}} \cdot \mathbf{C}_{\mathbf{i}}$ in formula (43) to an upper triangular matrix $\mathbf{U}_{\mathbf{i}}$ of the order of 3 had been proposed by the following way. We arrange the elements of the matrix $\frac{\mathbf{1}}{\mu_{\mathbf{0}}^{\mathbf{2}}} \cdot \mathbf{C}_{\mathbf{i}}$ in turn by columns in array **C** of length 6. After the performance of operations sequentially by the below represented procedure:

$$\mathbf{C(6) = 1/\sqrt{C(6)};\quad C(5) = C(5).C(6);\quad C(4) = C(4).C(6);}$$

$$\mathbf{C(3) = \frac{1}{\sqrt{C(3) - C(5)^2}};\quad C(2) = \left[C(2) - C(4).C(5)\right].C(3);}$$

$$\mathbf{C(1) = \frac{1}{\sqrt{C(1) - C(2)^2 - C(4)^2}};\quad C(2) = -C(1).C(2).C(3);}$$

$$\mathbf{C(4) = -\left[C(1).C(4) + C(2).C(5)\right].C(6);\quad C(5) = -C(3).C(5).C(6),}$$

we will obtain corresponding elements of the upper triangular matrix $\mathbf{U}_{\mathbf{i}}$ arranged by columns.

Before the separate adjustment of the GNSS network, we ought to choose one GNSS point to be "a fixed point" that has spatial coordinates in both the ITRF and the NSRS. Without losing generality, this fixed point is numbered with the number sign 1. Based on a method of a temporary fixation of an initial point, proposed in [18], an inverse weight matrix $\overline{\mathbf{Q}}_{\mathbf{F}}$ of the spatial coordinates of the fixed point in the ITRF is accepted to be $\mathbf{10^{-2m}.E_{3x3}}$, that is.

$$\overline{\mathbf{Q}}_{\mathbf{F}} = \mathbf{10^{-2m}.E_{3x3}}, \tag{45}$$

where number m is equal to 6, $\mathbf{E}_{\mathbf{3x3}}$-unit matrix of the order of 3 × 3.

The choice of a fixed point guarantees the nonsingularity of normal matrix obtained in a process of the adjustment of the GNSS network. Below, we will prove that after the combined adjustment of terrestrial geodetic and GNSS networks, the temporary fixation of the initial point will be automatically eliminated.

To start the separate adjustment of the GNSS network in the ITRF, on an account of formula (45), we obtain an initial upper triangular matrix $(\mathbf{T_S})_0$ of the order of K for the recurrent adjustment process in the following form:

$$(\mathbf{T_S})_0 = \begin{bmatrix} \mathbf{10^m.E_{3x3}} & \mathbf{0} \\ \mathbf{0} & \mathbf{10^{-m}.E_{(K-3)x(K-3)}} \end{bmatrix}.$$

The recurrent adjustment process will be realized by the T- recurrent algorithm sequentially for every observation equation from the system of observation equations (44). The outlier detection will be performed if the *i*th vector of pseudo-observations $\overline{\mathbf{Y}}_\mathbf{i}$ (41) is redundant.

After the accomplishment of the separate adjustment of the GNSS network with the insertion of all N vectors of pseudo-observations in the form (41) in the recurrent adjustment process by the T-recurrent algorithm, if outliers are encountered in the network, we will perform outlier detection using the method represented in Subsection 2.3.

If the GNSS network does not contain outliers, the obtained upper triangular matrix $\mathbf{T_S} = (\mathbf{T_S})_\mathbf{N}$ of the order of K will be related to the normal matrix $\mathbf{R_S}$ in the system of observation equations (17) by the formula $\mathbf{R_S} = \mathbf{T_S^T.T_S}$. Therefore for the combined adjustment of the terrestrial geodetic and GNSS networks with the solving of the system of observation equations (17) by the T-recurrent algorithm, second subsystem of observation equations in (17) will be expressed in the form:

$$\overline{\mathbf{V}}_\mathbf{S}^\mathbf{Kx1} = \mathbf{T_S}.\boldsymbol{\delta}\hat{\boldsymbol{\tau}}_\mathbf{S}^\mathbf{Kx1} - \overline{\boldsymbol{\Omega}}.\delta\boldsymbol{\omega}^\mathbf{3x1} + \overline{\mathbf{L}}_\mathbf{S}^\mathbf{Kx1}, \tag{46}$$

where $\overline{\mathbf{V}}_\mathbf{S}^\mathbf{Kx1} = \mathbf{T_S}.\mathbf{V}_\mathbf{S}^\mathbf{Kx1}$; $\overline{\boldsymbol{\Omega}} = \mathbf{T_S}.\boldsymbol{\Omega}$; $\overline{\mathbf{L}}_\mathbf{S} = \mathbf{T_S}.\mathbf{L_S}$; $\mathbf{T_S}$ is the upper triangular matrix obtained from the separate adjustment of the GNSS network in the ITRF.

The usage of the T-recurrent algorithm for solving the system of observation equations (17) has the remarkable advantage of being very simple for solving the subsystem of observation equations (46), created based on the transformation of the results of the separate adjustment of the GNSS network from the ITRF into the NSRS.

The subsystem of observation equations (46) has a unit weight matrix $\overline{\mathbf{P}}_\mathbf{S} = \mathbf{E_{KxK}}$ of the order of K. To start the combined adjustment of the terrestrial geodetic and GNSS networks in the NSRS by the T-recurrent algorithm, we obtain an initial upper triangular matrix $\mathbf{T_0}$ with the order of K + 3 of the recurrent adjustment process in the following form:

$$\mathbf{T_0} = \begin{bmatrix} \mathbf{T_\tau^{kxk}} & \mathbf{0} \\ \mathbf{0} & \mathbf{10^{-m}.E_{\overline{K}x\overline{K}}} \end{bmatrix}, \tag{47}$$

where an upper triangular matrix $\mathbf{T_\tau^{kxk}}$ is related to weight matrix $\mathbf{P_\tau}$ of the first subsystem of observation equations in (17) by the formula $\mathbf{P_\tau} = \mathbf{T_\tau^T.T_\tau}$ order $\overline{\mathbf{K}} = \mathbf{K} + \mathbf{3} - \mathbf{k}.$

The task of the combined adjustment of the terrestrial geodetic and GNSS networks in the NSRS will be performed by the T-recurrent algorithm based on a sequential insertion of observation equations from the subsystem of observation equations (46) in the recurrent adjustment process with the usage of the initial matrix $\mathbf{T_0}$(47). Because the outlier detection in the GNSS network had been performed in the process of the separate adjustment of this network, then in the process of solving the abovementioned task, the outlier detection will be performed for the data of terrestrial geodetic network. The results of the combined adjustment of the terrestrial geodetic and GNSS networks in the NSRS will be performed by the T-recurrent algorithm determined by the formulas (21), (22), (24) represented in Subsection 2.2.

For the end of this subsection, we prove that performing the separate adjustment of the GNSS network in the ITRF, the temporary fixation of an initial point by assigning the inverse matrix $\overline{\mathbf{Q}}_{\mathbf{`F}}$(45) to the spatial coordinates of the fixed point will be automatically eliminated after the combined adjustment of the terrestrial geodetic and GNSS networks.

It is assumed that for all N baseline vectors in the GNSS network, a system consisting of 3.N observation equations has been created in the following form:

$$\mathbf{V} = \mathbf{A}.\delta\mathbf{H} + \mathbf{L}^{\mathbf{(0)}} \tag{48}$$

with weight matrix **P**.

Solving the system of observation equations (48) under condition $\mathbf{V^TPV} = \mathbf{min}$, we obtain a normal matrix $\hat{\mathbf{R}}_\mathbf{S} = \mathbf{A^TPA}$. If in the GNSS network there is not any fixed point, that is, the GNSS network becomes the free network, then the normal matrix $\hat{\mathbf{R}}_\mathbf{S}$ will be singular due to the rank defect d = 3. In this case, the matrix of coefficients $\mathbf{A}$ with dimension 3.N × K has the rank defect d = 3 and satisfies the condition:

$$\mathbf{A}.\mathbf{\Omega} = \mathbf{0}, \tag{49}$$

where matrix $\mathbf{\Omega}$ has the form (16) with K = 3.NP rows and 3 columns.

For the strict separate adjustment of the GNSS network in the ITRF and avoiding the singularity of the normal matrix $\hat{\mathbf{R}}_\mathbf{S}$, on an account of the formula (45), we performed the above represented method of the temporary fixation of initial point with an additional usage of system of observation equations $\mathbf{V_F} = \mathbf{E_1}.\delta\mathbf{H}_1$ to which the weight matrix $\mathbf{P_F} = \overline{\mathbf{Q}}_\mathbf{F}^{-1} = \mathbf{10^{2m}}.\mathbf{E_{3x3}}$ has been assigned, where $\mathbf{E_1}$ is the unit matrix of the order of 3; $\boldsymbol{\delta}\mathbf{H_1}$ is the subvector of corrections to the spatial coordinates of the fixed point with number sign 1 of the GNSS network. In this case the separate adjustment of the GNSS network in the ITRF will be accomplished based on simultaneous solving the above mentioned system of observation equations with the system of observation equations (48) under the condition $\mathbf{V_F^T P_F V_F} + \mathbf{V^T PV} = \mathbf{min}$. As a result, we obtain the normal matrix.

$$\mathbf{R_S} = \hat{\mathbf{R}}_\mathbf{S} + \hat{\mathbf{P}}_\mathbf{F}, \tag{50}$$

where the matrix $\hat{\mathbf{P}}_\mathbf{F}$ has the form:

$$\hat{\mathbf{P}}_\mathbf{F} = \begin{bmatrix} \mathbf{P_F^{3x3}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{\mathbf{KxK}} \tag{51}$$

As mentioned in Subsection 2.1, the normal matrix $\mathbf{R_S}$ (50) is used as the weight matrix $\mathbf{P_S}$ assigned to the second subsystem of observation equations in (17).

On an account of (49), the product $\hat{\mathbf{R}}_\mathbf{S}.\boldsymbol{\Omega} = \mathbf{0}$. When we get relationship from (50):

$$\mathbf{R_S}.\boldsymbol{\Omega} = \hat{\mathbf{P}}_\mathbf{F}.\boldsymbol{\Omega} = \begin{bmatrix} \mathbf{P_F} & \mathbf{0} & \mathbf{...0} \end{bmatrix}^\mathbf{T}. \tag{52}$$

Therefrom we infer the equality:

$$\boldsymbol{\Omega}^\mathbf{T}.\mathbf{R_S}.\boldsymbol{\Omega} = \boldsymbol{\Omega}^\mathbf{T}.\hat{\mathbf{P}}_\mathbf{F}.\boldsymbol{\Omega} = \mathbf{P_F} \tag{53}$$

Now performing the combined adjustment of the terrestrial geodetic and GNSS networks in the NSRS with solving the system of observation equations (17) under the condition $\mathbf{V_\tau^T.P_\tau.V_\tau} + \mathbf{V_S^T.P_S.V_S} = \mathbf{V_\tau^T.P_\tau.V_\tau} + \mathbf{V_S^T.R_S.V_S} = \mathbf{min}$, where normal matrix $\mathbf{R_S}$ has the form (50), we obtain a system of normal equations in the following form:

$$\begin{bmatrix} \boldsymbol{\Omega}^\mathbf{T}.\mathbf{R_S}.\boldsymbol{\Omega} & \vdots & -\boldsymbol{\Omega}^\mathbf{T}.\mathbf{R_S} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ -\mathbf{R_S}.\boldsymbol{\Omega} & \vdots & \hat{\mathbf{P}}_\boldsymbol{\tau} + \mathbf{R_S} \end{bmatrix} . \begin{bmatrix} \boldsymbol{\delta\omega} \\ \cdots \\ \boldsymbol{\delta}\hat{\boldsymbol{\tau}}_\mathbf{S} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{\Omega}^\mathbf{T}.\mathbf{R_S}.\mathbf{L_S} \\ \cdots\cdots \\ \mathbf{R_S}.\mathbf{L_S} \end{bmatrix} = \mathbf{0} \tag{54}$$

Additionally, the matrix $\hat{\mathbf{P}}_\boldsymbol{\tau}$ has the form:

$$\hat{\mathbf{P}}_{\tau} = \begin{bmatrix} \mathbf{P}_{\tau}^{\mathbf{kxk}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{\mathbf{KxK}} .$$

For the system of normal equations (54), substituting $\boldsymbol{\delta\omega} = \left(\boldsymbol{\Omega}^{\mathbf{T}}.\mathbf{R}_{\mathbf{S}}.\boldsymbol{\Omega}\right)^{-1}.\boldsymbol{\Omega}^{\mathbf{T}}.\mathbf{R}_{\mathbf{S}}.(\boldsymbol{\delta}\hat{\boldsymbol{\tau}}_{\mathbf{S}} + \mathbf{L}_{\mathbf{S}})$ inferred from the first subsystem of normal equations into the second subsystem of normal equations, we will obtain a transformed system of normal equations in the form:

$$\left[\hat{\mathbf{P}}_{\tau} + \mathbf{R}_{\mathbf{S}} - \mathbf{R}_{\mathbf{S}}.\boldsymbol{\Omega}\left(\boldsymbol{\Omega}^{\mathbf{T}}.\mathbf{R}_{\mathbf{S}}.\boldsymbol{\Omega}\right)^{-1}.\boldsymbol{\Omega}^{\mathbf{T}}.\mathbf{R}_{\mathbf{S}}\right].\boldsymbol{\delta}\hat{\boldsymbol{\tau}}_{\mathbf{S}} + \left[\mathbf{R}_{\mathbf{S}} - \mathbf{R}_{\mathbf{S}}.\boldsymbol{\Omega}\left(\boldsymbol{\Omega}^{\mathbf{T}}.\mathbf{R}_{\mathbf{S}}.\boldsymbol{\Omega}\right)^{-1}.\boldsymbol{\Omega}^{\mathbf{T}}.\mathbf{R}_{\mathbf{S}}\right].\mathbf{L}_{\mathbf{S}} = \mathbf{0}. \quad (55)$$

On an account of the formulas (16), (50), (51), (52), (53) we obtain:

$$\mathbf{R}_{\mathbf{S}} - \mathbf{R}_{\mathbf{S}}.\boldsymbol{\Omega}\left(\boldsymbol{\Omega}^{\mathbf{T}}.\mathbf{R}_{\mathbf{S}}.\boldsymbol{\Omega}\right)^{-1}.\boldsymbol{\Omega}^{\mathbf{T}}.\mathbf{R}_{\mathbf{S}} = \hat{\mathbf{R}}_{\mathbf{S}} + \hat{\mathbf{P}}_{\mathbf{F}} - \hat{\mathbf{P}}_{\mathbf{F}}.\boldsymbol{\Omega}.\mathbf{P}_{\mathbf{F}}^{-1}.\boldsymbol{\Omega}^{\mathbf{T}}.\hat{\mathbf{P}}_{\mathbf{F}}^{\mathbf{T}} = \hat{\mathbf{R}}_{\mathbf{S}} + \hat{\mathbf{P}}_{\mathbf{F}} - \hat{\mathbf{P}}_{\mathbf{F}} = \hat{\mathbf{R}}_{\mathbf{S}}. \quad (56)$$

Finally, substituting (56) into (55), we obtain the following system of normal equations:

$$\left(\hat{\mathbf{P}}_{\tau} + \hat{\mathbf{R}}_{\mathbf{S}}\right).\boldsymbol{\delta}\hat{\boldsymbol{\tau}}_{\mathbf{S}} + \hat{\mathbf{R}}_{\mathbf{S}}.\mathbf{L}_{\mathbf{S}} = \mathbf{0},$$

in which the effect of the temporary fixation of an initial point, made in the process of the separate adjustment of the GNSS network in the ITRF, fully has been eliminated.

It can be concluded that the usage of the method of the temporary fixation of initial point for the strict separate adjustment of the GNSS network in the ITRF and avoiding the singularity of the normal matrix $\hat{\mathbf{R}}_{\mathbf{S}}$ does not cause any influence on the results of the combined adjustment of the terrestrial geodetic and GNSS networks in the NSRS. Moreover, this method allows the spatial coordinates of the initial point be corrected after the abovementioned combined adjustment. We will lose valuable priori information regarding the spatial coordinates of the initial point of the GNSS network for the accuracy improvement of the national spatial coordinates of GNSS points in the NSRS, if the spatial coordinates of the abovementioned initial point of the GNSS network are considered to be nonerroneous.

3. Experimental results

3.1. Data

In [22], the results of the construction of the initial national spatial reference system VN2000–3D on the base of the orientation of the WGS84 ellipsoid to best fit it to the Hon Dau local quasigeoid at tide gauge Hon Dau with using the most stable 164 colocated GPS observations performed at the first- and second-order benchmarks had been presented. The GPS data had been processed in the ITRF2008 in the period 2009–2010. The coordinate transformation parameters from the ITRF to the VN2000–3D have the following values:

$$X_0 = 204{,}511083\,m,\quad Y_0 = 42{,}192468\ m,\quad Z_0 = 111{,}417880\ m,$$

$$\varepsilon_X = -0''.011168229,\ \varepsilon_Y = 0''.085600577,\ \varepsilon_Z = -0''.400462723,\ \Delta m = 0.$$

In [24], the results of the construction of the initial national quasigeoid model VIGAC2017 with the accuracy level of ±5.8 cm had been presented.

From 11 to 14 November 2013, Vietnam Institute of Geodesy and Cartography (VIGAC) had accomplished four sessions of 24 h GPS observations at 11 points of the GPS network in the North Vietnam (see **Figure 1**). Average distance between GPS points is 105 km. The GPS data had been processed in the ITRF2008 by the software Bernese v. 5.2 using IGS service products.

The GPS network has five common (ground control) points C052, C022, C045, C033, C004, that have the approximate national spatial coordinates in VN2000–3D (see **Table 1**) and have been numbered sequentially from 1 to 5. In Vietnam, horizontal coordinates of geodetic points are determined in VN2000-2D, and their normal heights are determined in national the vertical reference system Haiphong1972 (HP72). On an account of the national quasigeoid model VIGAC2017, the RMS of the national ellipsoidal coordinates of the geodetic points had been considered equal to $m_B = 0''.002;\ m_L = 0''.0015;\ m_H = 0.097\ m$ After expressing the $m_B,\ m_L$ in the radian unit, we had created the variance–covariance matrix $\mathbf{K}^{3x3}_{B,L,H}$, that is considered equivalent to the abovementioned five common points. From that for every common point, we had created the variance–covariance matrix $\mathbf{K}^{3x3}_{XYZ} = \chi.\mathbf{K}^{3x3}_{B,L,H}.\chi^{T}$, where.

$$\chi = \begin{bmatrix} -(\rho_M + H).\sin B.\cos L & -(\rho_N + H).\cos B.\sin L & \cos B.\cos L \\ -(\rho_M + H).\sin B.\sin L & (\rho_N + H).\cos B.\cos L & \cos B.\sin L \\ (\rho_M + H).\cos B & 0 & \sin B \end{bmatrix},$$

ρ_M is the radius of curvature in the meridian plane; ρ_N is the radius of curvature in the first vertical plane.

On the basis of the algorithm of transformation of the variance-covariance matrix to the upper triangular matrix, represented in Subsection 2.4, we had got the upper triangular matrices for five common points in the NSRS in the following forms:

$$T_1 = \begin{bmatrix} 0.112997 & -0.291820 & -0.678800 \\ 0 & 0.057843 & 0.179161 \\ 0 & 0 & 0.074068 \end{bmatrix}; \quad T_2 = \begin{bmatrix} 0.112708 & -0.2999733 & -0.713919 \\ 0 & 0.057491 & 0.182105 \\ 0 & 0 & 0.074403 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 0.112522 & -0.287024 & -0.680706 \\ 0 & 0.057661 & 0.182140 \\ 0 & 0 & 0.074355 \end{bmatrix}; \quad T_4 = \begin{bmatrix} 0.113212 & -0.306647 & -0.716235 \\ 0 & 0.057659 & 0.179026 \\ 0 & 0 & 0.074115 \end{bmatrix}$$

$$T_5 = \begin{bmatrix} 0.113795 & -0.325441 & -0.750297 \\ 0 & 0.057674 & 0.176483 \\ 0 & 0 & 0.073903 \end{bmatrix}.$$

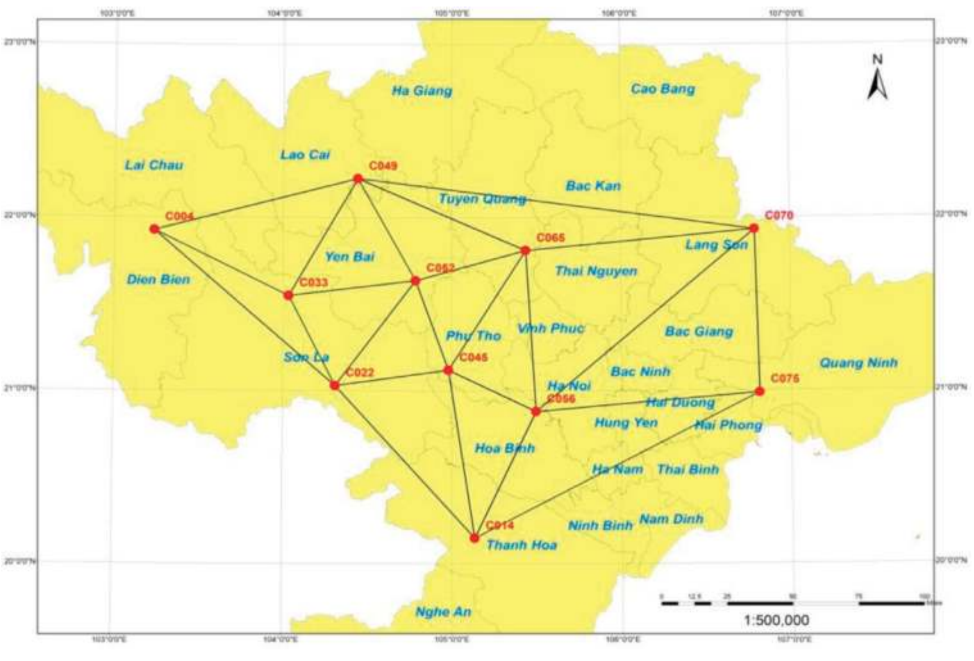

Figure 1. The GPS network in the North Vietnam.

No	Common (ground control) points	Approximate spatial coordinates in VN2000–3D		
		$X_\tau^{(0)}$ (m)	$Y_\tau^{(0)}$ (m)	$Z_\tau^{(0)}$ (m)
1	C052	−1513714.136	5735121.344	2337092.916
2	C022	−1472179.244	5771490.833	2274632.893
3	C045	−1538604.244	5750184.813	2283824.080
4	C033	−1439254.798	5758082.515	2328258.441
5	C004	−1355466.287	5762595.502	2367026.391

Table 1. Approximate national spatial coordinates of the ground control points C052, C022, C045, C033, C004 in VN2000–3D.

These upper triangular matrices will be used for creating the submatrix T_τ^{15x15} in the initial upper triangular matrix T_0 in the form (36) with the purpose of the combined adjustment of the GPS network, shown in **Figure 1**, into VN2000–3D.

3.2. Results

In [28], the experiments of the combined adjustment of the GPS network, shown in **Figure 1**, in VN2000–3D had been accomplished. The GPS network had been adjusted separately in the ITRF2008 by the T-recurrent algorithm with the temporary fixation of an initial point for GPS point C052. The adjusted spatial coordinates of all 11 GPS points had been transformed from the ITRF2008 to VN2000–3D (see **Table 2**).

The last spatial coordinates of all 11 GPS points in VN2000–3D obtained after the combined adjustment of the GPS network in VN2000–3D based on insertion of the system of observation equations in the recurrent adjustment process by the T–recurrent algorithm are shown in **Table 3**.

No	Points	$X_\theta^{(0)}$ (m)	$Y_\theta^{(0)}$ (m)	$Z_\theta^{(0)}$ (m)
1	C052	−1513714.080	5735121.312	2337092.905
2	C022	−1472179.140	5771490.861	2274632.888
3	C045	−1538604.194	5750184.888	2283824.115
4	C033	−1439254.730	5758082.565	2328258.478
5	C004	−1355466.208	5762595.543	2367026.437
6	C049	−1473387.470	5720475.157	2397685.449
7	C065	−1576880.962	5710639.604	2355075.723
8	C056	−1592782.951	5745126.896	2259055.940
9	C014	−1564014.757	5782717.956	2183131.028
10	C075	−1723353.397	5702825.750	2270215.032
11	C070	−1710134.998	5667162.050	2367393.077

Table 2. Spatial coordinates of all 11 GPS points had been transformed from the ITRF2008 to VN2000–3D.

No	Points	$\tilde{X}$ (m)	$\tilde{Y}$ (m)	$\tilde{Z}$ (m)
1	C052	−1513714.150	5735121.372	2337092.873
2	C022	−1472179.207	5771490.916	2274632.850
3	C045	−1538604.253	5750184.910	2283824.046
4	C033	−1439254.784	5758082.567	2328258.392
5	C004	−1355466.267	5762595.567	2367026.370
6	C049	−1473387.532	5720475.185	2397685.386
7	C065	−1576881.025	5710639.642	2355075.670
8	C056	−1592783.012	5745126.934	2259055.888
9	C014	−1564014.818	5782717.991	2183130.973
10	C075	−1723353.458	5702825.780	2270214.971
11	C070	−1710135.062	5667162.086	2367393.020

Table 3. Final spatial coordinates of all 11 GPS points in VN2000–3D after the combined adjustment of the GPS network.

The mean values of the RMS of national ellipsoidal coordinates of GPS points after solving the task of the combined adjustments of the GPS network in VN2000–3D are equal to $m_{\tilde{B}} = 0''.0007;\ m_{\tilde{L}} = 0''.0005;\ m_{\tilde{H}} = 0.023\ m.$ That confirmed the significant improvement of positional accuracy of the GNSS points in the NSRS after solving of the task of the combined adjustments of the GNSS network in the NSRS.

4. Conclusions

A tendency of construction of the NSRS strongly is promoted in many countries in the world due to development of the passive GNSS networks, comprising the ground control points and some CORS stations, based on the GNSS methods and results of building of the highly accurate national geoid/quasigeoid models at the centimeter level of accuracy thanks to detailed gravimetric data and the Earth gravitational models with high resolution.

From demands of usage of the high accurate spatial coordinates of GNSS points in the ITRF for different geodetic applications and next their usage for the construction of the national spatial reference frame has been arisen techno-scientific task of the separate adjustment of the passive GNSS network in the ITRF and next its combined adjustment with the terrestrial geodetic network in the NSRS.

In this chapter, a recurrent adjustment method with Givens rotation had been represented for solving the above mentioned task on an account of its abilities to use the technique of sparse matrix, to detect outliers in the recurrent adjustment process and to find them, especially to use effectively results of the separate adjustment of the passive GNSS network in the ITRF for creating the system of observation equations (46) and its realization in the process of the combined adjustment of the passive GNSS network with the terrestrial geodetic network in the NSRS.

In this chapter, the method of the temporary fixation of an initial point used for the separate adjustment of the passive GNSS network in the ITRF had been represented. The abovementioned temporary fixation of an initial point allows not only to perform the strict adjustment of the passive GNSS network in the ITRF and to avoid the singularity of transformed matrix but also to correct the spatial coordinates of fixed point after the combined adjustment of the GNSS network in the NSRS. Additionally, the temporary fixation of the initial point does not cause any influence to the results of the above represented combined adjustment.

The results of experiments performed on the basis of the usage of the T-recurrent algorithm for the separate adjustment of the GPS network in the North Vietnam and the its combined adjustment into VN2000–3 D confirmed the significant improvement of positional accuracy of the GPS points in VN2000–3 D and effectivity of the T-recurrent algorithm in mathematical processing of the GPS network for the construction of the national spatial reference frame. Apart from that, after the combined adjustment of the GPS network in VN2000–3 D, the horizontal and vertical position accuracy of the GPS points had reached the few centimeter level. The mean values of the RMS of national ellipsoidal coordinates of GPS points after solving task of the combined adjustments of the GPS network in VN2000–3D are equal to $m_{\tilde{B}} = 0''.0007;\ m_{\tilde{L}} = 0''.0005;\ m_{\tilde{H}} = 0.023\ m$ ./.

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Author details

Ha Minh Hoa

Address all correspondence to: minhhoavigac@gmail.com

Vietnam Institute of Geodesy and Cartography, Hanoi, Vietnam

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